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Quenched n-Vector p-Spin Model

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A disordered n-vector model with p spin interactions previously introduced is studied for the quenched case by means of the replica method and a generalized Parisi theory. We present formal solutions for general n and p and then study the case $p \to \infty$. The high-temperature solution is stable at all temperatures and there is only one phase transition at a temperature T_g. Only longitudinal low-temperature solutions are possible. There is one spin-glass solution, and it is stable for all $T < T_g$. The phase transition at T_g is of first order and displays a jump discontinuity in the order parameters $q_j^{(L)}$ and d. The spin-glass free energy is temperature dependent for $n > 1$ while it is constant when $n = 1$.

KEY WORDS: n-vector model; random spin models; spin glass; Parisi theory.

1. INTRODUCTION

In a previous paper,[1] we presented a generalization of the Stanley n-vector model with infinite-range potential[2,3] by introducing Gaussian random bonds and p spin interactions. The model is defined by a generalized Hamiltonian

$$-\beta\mathscr{H} \equiv \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1 \cdots i_p} \sum_{\alpha=1}^{n} S_{i_1}^{\alpha} \cdot \cdots \cdot S_{i_p}^{\alpha} \tag{1.1}$$

where the $\mathbf{S}_i^T \equiv (S_i^\alpha) \equiv (S_i^1,..., S_i^n)$ are classical n-vectors normalized to $\|\mathbf{S}_i\| = 1$, α, β denote running indices for the vector components, i denotes lattice sites, and $\mathbf{S}^T$ is the transposed vector $\mathbf{S}$. We have chosen the vector normalization $\|\mathbf{S}_i\| = 1$ in contrast to $\|\mathbf{S}_i\| = \sqrt{n}$ in ref. 1 in order to regularize the limit $p \to \infty$ in Section 6. The coupling constants $J_{i_1 \cdots i_p}$ are

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independent random variables with an appropriately scaled Gaussian distribution so as to give rise to an intensive free energy per spin,

$$P(J_{i_1 \cdots i_p}) \equiv \left[\frac{N^{p-1}}{\pi p! \, (\Delta J)^2} \right]^{1/2} \exp\left[- \frac{(J_{i_1 \cdots i_p})^2 \, N^{p-1}}{p! \, (\Delta J)^2} \right], \qquad \Delta J \equiv \beta \, \Delta \tilde{J} \tag{1.2}$$

$\Delta \tilde{J}$ represents the width of the Gaussian distribution, which for simplicity is assumed to be centered at $J_0 = 0$. The case of a nonzero mean can be treated in a canonical fashion.

For $n > 1$ and $p = 2$, the model is isotropic with a continuous $O(n)$ symmetry. For $p > 2$, anisotropy is introduced by replacing the continuous $O(n)$ symmetry with a discrete S_n symmetry.

For $n = 1$ our model represents the random Curie–Weiss model with p spin interactions which was introduced by Derrida.[4–8] For $p = 2$, the model becomes the random Stanley model with infinite-range interactions. It was first considered for $n = 3$ by Edwards and Anderson[9] and for $n = 2$ by Kirkpatrick and Sherrington.[10] The case of general n was first presented by Gabay and Toulouse[11–17] (see ref. 18 for a recent review). For $n = 2$ and general p we obtain the random planar rotator model with p spin interactions. For $n = 3$ and general p we have the random classical Heisenberg model with p spin interactions. All of these models have well-known submodels, e.g., the Sherrington–Kirkpatrick model for $n = 1$ and $p = 2$ and the random energy model for $n = 1$ and $p \to \infty$. However, we do not recover the random spherical model for $p = 2$ and $n \to \infty$, since this would require $n = N$ and hence a different limiting procedure and scaling.

In ref. 1 we proceeded to investigate the model (1.1) for the annealed case. It turned out that already the annealed model displays an unexpected richness of solutions and subtleties regarding their stability. We presented complete solutions for the cases $n = 2$ and $n = 3$. For general n, we managed to derive explicit forms of the order parameter equations and the free energy for the stable solutions of the model. These can be expressed in terms of hypergeometric functions ${}_1F_1$. The model is described by one order parameter μ_1. For all n and p there is one stable high-temperature phase and one stable low-temperature phase. The phase transition is of first order. For $n = 2$, it is continuous in the order parameters for $p \leqslant 4$ and has a jump discontinuity in the order parameters if $p > 4$. For $n = 3$, it has a jump discontinuity in the order parameters for all p.

In this paper, we shall consider the model (1.1) for the case of quenched random couplings. While the case of general n and p seems to be quite involved, there are three limiting cases worth considering: (1) $n = 1$ and general p, (2) general n and $p = 2$, and (3) general n and $p \to \infty$. The first two cases have been treated in the literature and we describe briefly the main features below. In this paper, we shall consider the case of general n and $p \to \infty$.

For quenched random spin systems, even mean-field theory has proven to be very subtle. The first infinite-range Ising spin-glass model was proposed by Sherrington and Kirkpatrick (SK).[19] In 1980 Derrida[4,5] showed that the SK model could be generalized to models involving p spin interactions and that in the limit of $p \to \infty$ they simplified to a random energy model, which consists of a collection of independently distributed random energy levels. He was then able to solve this model without recourse to the replica trick. Gross and Mézard[6] confirmed his results for the same $p \to \infty$ model by using the replica method and Parisi's replica-symmetry-breaking scheme. Gardner[7] and Stariolo[8] studied the model for finite p. They found that for $p = 2$ and $p = \infty$ there are two phases, a high-temperature phase above a critical temperature T_c and a spin-glass phase below T_c. The phase transition is of second order and continuous in the order parameter $q(x)$ for $p = 2$, but has a jump discontinuity in the order parameter for $p = \infty$. For all finite $p > 2$ there are three phases, (1) a high-temperature phase above a critical temperature T_{c1}, (2) a spin-glass phase SG1 which is stable between T_{c1} and a second critical temperature $T_{c2} < T_{c1}$, and (3) a spin-glass phase SG2 below T_{c2}. The phase transition at T_{c1} is of second order with no latent heat, but displays a jump discontinuity in the order parameter. The phase transition at T_{c2} is of second order and continuous in the order parameter. Although a stability analysis shows that the disordered high-temperature solution is stable at all temperatures, its entropy becomes negative at some temperature $T' < T_{c1}$. This suggests that replica symetry is broken. By performing the first step in Parisi's replica-symmetry-breaking scheme, one obtains the spin-glass phase SG1. The nature of the spin-glass phase SG2, however, is not completely understood, since the full replica-symmetry-breaking scheme would have to be performed in this case.

As we mentioned before, the random Stanley model with infinite-range interactions given by $p = 2$ and arbitrary n represents the second limiting case of our model. The replica-symmetric theory for the corresponding quenched problem was first presented by Gabay and Toulouse[11] and later extended by Cragg *et al.* to include a stability analysis with respect to replica symmetry breaking[12] and anisotropic interactions.[13] The parameter q from Ising spin glasses which describes the overlap between pure states of the model generalizes to a matrix parameter $(q^{(L)}, q^{(T)},..., q^{(T)})$ with the longitudinal and transverse components $q^{(L)}$ and $q^{(T)}$ being different in the presence of a magnetic field or some anisotropic interaction. In addition, a third nontrivial spin-glass parameter d (called the quadrupolar deformation parameter), which describes the self-correlation (3.3) of a pure state of the model, has to be introduced. Just as for the Ising case, only the high-temperature solution is replica-symmetric, while a Parisi-type solution with broken replica symmetry and corresponding Parisi functions $q^{(L)}(x)$ and $q^{(T)}(x)$ describes the low-temperature phases.[14–18] In a magnetic field $\mathbf{H}$, a replica-symmetric longitudinal solution $q^{(T)}=0$ exists above an instability line in H–T space called the Gabay–Toulouse (GT) line.[11] The GT line replaces the de Almeida–Thouless (AT) line[20] from the SK model. Below the GT line characterized by T_{c1}, the replica-symmetric solution becomes unstable and freezing of transverse components $q^{(T)} \neq 0$ sets in combined with loss of time ergodicity, i.e., we have replica symmetry breaking. However, in a region $T_{c1} > T > T_{c2}$ the order parameters $q^{(L)}(x)$ and $q^{(T)}(x)$ are nearly constant, i.e., replica symmetry is only weakly broken. Below a second line characterized by T_{c2} and which has the same H, T dependence as the AT line, strong longitudinal symmetry breaking sets in. In contrast to the AT line, this second line is not an instability line, but rather a crossover line from weak to strong replica symmetry breaking. By introducing an anisotropic interaction $-D\mathbf{S}_{i_1}^T\mathbf{S}_{i_2}$ into the Hamiltonian as $H \to 0$, the system will settle into a longitudinal $(q^{(L)} \neq 0,\ q^{(T)}=0)$ or transverse $(q^{(L)}=0,\ q^{(T)} \neq 0)$ spin-glass phase depending on whether $D>0$ or $D<0$. For $D \sim 0$, the system can in addition occupy a mixed spin-glass phase $(q^{(L)} \neq 0,\ q^{(T)} \neq 0)$.[13,17]

As stated before, in this paper we investigate the model (1.1) for general n while $p \to \infty$. The paper is organized as follows. In Section 2 we derive formally the free energy and order parameter equations for arbitrary n and p by means of a generalized replica method. In Section 3 the connection between our replica formalism and the overlaps and self-correlations of the pure states of the model is established rigorously. In particular, we define probability distributions $P_{\alpha\beta}$ and $W_{\alpha\beta}$ which constitute the physical order parameters of the system and also state two intuitive distributions P and W which average out some of the information contained in $P_{\alpha\beta}$ and $W_{\alpha\beta}$. Using a different language and in a slightly less rigorous fashion, this relationship between the space of pure states and the replica formalism for $n>1$ has also been established independently in ref. 16. In Section 4 we generalize the Parisi theory to $n>1$. We derive the form of matrices $Q_{\rho\tau}$ which arise in our replica formalism and we describe the geometrical degeneracy of the solutions. We then state the high-temperature solution and give a precise description of the theory of replica symmetry breaking for $n>1$. In particular, we prove by means of the Hölder inequality that replica symmetry for the diagonal (quadrupolar, self-correlation) parameters d_ρ is always conserved. Finally, we give a formal expression for the free energy of our model after k steps of symmetry breaking for general n and p. In the case of $p=2$, this overlaps largely with the generalized

Parisi theory presented independently in refs. 11–17. However, the exact proof that replica symmetry for the self-correlation (quadrupolar) parameters d_ρ can never be broken and the discussion of the geometrical degeneracy of the solutions still represent new contributions for $p=2$. In Section 5 we investigate replica-symmetric solutions as $p \to \infty$. We find that only longitudinal solutions are possible, and show the stability of the high-temperature solution. In Section 6 we derive the low-temperature solution as $p \to \infty$. The procedure for replica symmetry breaking terminates after the first step just as for the $n=1$ model,[6] and again, only longitudinal solutions are possible. We obtain an analytic expression for the spin-glass transition temperature T_g and show that there exists only one spin-glass solution and that it is stable for all $T < T_g$.

2. QUENCHED ORDER PARAMETER EQUATIONS

In order to perform the average over the quenched random couplings, we use the replica trick first introduced by Edwards and Anderson.[9] However, to avoid confusion with the symbol n for the dimensionality of the vectors $\mathbf{S}$, we denote the number of replicas by r rather than by n as is the usual convention. That is, we obtain the quenched free energy $\bar{A}$ by using the relation

$$-\beta \bar{A} = \overline{\ln Z_N} = \lim_{r \to 0} \frac{1}{r} (\overline{Z_N^r} - 1) \tag{2.1}$$

where Z_N is the partition function of the system and the bar $\overline{\dots}$ denotes the average over the quenched random couplings. r is the number of replicas, which we take initially to be an integer $\geqslant 1$ and then analytically continue to $r=0$.

From Eqs. (1.1) and (1.2) we have

$$\begin{aligned}
\overline{Z_N^r} = & \int_{-\infty}^{\infty} \prod_{1 < i_1 < \cdots < i_p \leqslant N} P(J_{i_1 \cdots i_p})\, dJ_{i_1 \cdots i_p} \\
& \times \left[\mathrm{Tr}_{\{\mathbf{S}_i\}} \exp\left(\sum_{\alpha=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1 \cdots i_p} S_{i_1}^{\alpha} \cdot \cdots \cdot S_{i_p}^{\alpha} \right) \right]^r \\
= & \int_{-\infty}^{\infty} \prod_{1 \leqslant i_1 < \cdots < i_p \leqslant N} P(J_{i_1 \cdots i_p})\, dJ_{i_1 \cdots i_p} \\
& \times \mathrm{Tr}_{\{{}^{\sigma}\mathbf{S}_i\}} \exp\left(\sum_{\rho=1}^{r} \sum_{\alpha=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1 \cdots i_p}\, {}^{\rho}S_{i_1}^{\alpha} \cdots {}^{\rho}S_{i_p}^{\alpha} \right)
\end{aligned} \tag{2.2}$$

where the indices ρ, τ, and σ refer to replicas. Evaluating the Gaussian integral in (2.2) gives

$$\overline{Z_N^r} = \mathrm{Tr}_{\{{}^\sigma\mathbf{s}_i\}} \exp\left[\frac{p!\,(\Delta J)^2}{4N^{p-1}} \sum_{1\leqslant i_1<\cdots<i_p\leqslant N} \left(\sum_{\rho=1}^{r}\sum_{\alpha=1}^{n} {}^\rho S_{i_1}^\alpha \cdots {}^\rho S_{i_p}^\alpha\right)^2\right]$$
$$= \mathrm{Tr}_{\{{}^\sigma\mathbf{s}_i\}} \exp\left[\frac{(\Delta J)^2}{4N^{p-1}}\left(N^p \sum_{\rho,\tau=1}^{r}\sum_{\alpha,\beta=1}^{n} q_{\alpha\beta;\rho\tau}^p + O(N^{p-1})\right)\right] \tag{2.3}$$

where we have defined

$$q_{\alpha\beta;\rho\tau} \equiv \frac{1}{N}\sum_{i=1}^{N} {}^\rho S_i^\alpha\, {}^\tau S_i^\beta = O(1) \qquad \text{as} \quad N\to\infty \tag{2.4}$$

We evaluate the trace in Eq. (2.3) by introducing a Lagrange multiplier matrix $\lambda_{\alpha\beta;\rho\tau}$. In the limit of large N, we get

$$\overline{Z_N^r} \xrightarrow{N\to\infty} \int_{-\infty}^{\infty} \prod_{\alpha,\beta=1}^{n}\prod_{\rho,\tau=1}^{r} dq_{\alpha\beta;\rho\tau} \int_{-i\infty}^{i\infty} \prod_{\alpha,\beta=1}^{n}\prod_{\rho,\tau=1}^{r} \frac{d\lambda_{\alpha\beta;\rho\tau}}{2\pi}$$
$$\times \exp[NG(q_{\alpha\beta;\rho\tau}, \lambda_{\alpha\beta;\rho\tau})]\left(\frac{N}{2}\right)^{(n+r)^2} \tag{2.5}$$

where

$$G(q_{\alpha\beta;\rho\tau}, \lambda_{\alpha\beta;\rho\tau}) \equiv \frac{(\Delta J)^2}{4}\sum_{\alpha,\beta=1}^{n}\sum_{\rho,\tau=1}^{r} q_{\alpha\beta;\rho\tau}^p - \frac{1}{2}\sum_{\alpha,\beta=1}^{n}\sum_{\rho,\tau=1}^{r} \lambda_{\alpha\beta;\rho\tau} q_{\alpha\beta;\rho\tau}$$
$$+ \ln \mathrm{Tr}_{\{{}^\sigma\mathbf{s}\}} \exp\left(\frac{1}{2}\sum_{\alpha,\beta=1}^{n}\sum_{\rho,\tau=1}^{r} \lambda_{\alpha\beta;\rho\tau}\, {}^\rho S^\alpha\, {}^\tau S^\beta\right) \tag{2.6}$$

Equation (2.5) can then be evaluated by the method of steepest descent,

$$\overline{Z_N^r} \xrightarrow{N\to\infty} \exp[NG^*]\cdot C \tag{2.7}$$

where C is a constant independent of N and where G^* is the dominant saddle point of G. The quenched free energy per spin is then obtained from Eq. (2.1) as

$$\overline{\left(\frac{a}{kT}\right)} = \lim_{N\to\infty}\lim_{r\to 0}\left(-\frac{1}{N}\frac{\overline{Z_N^r}-1}{r}\right) = \lim_{r\to 0} -\frac{G^*}{r} \tag{2.8}$$

if the limit $N\to\infty$ can be taken before the limit $r\to 0$ (as is always assumed in the replica formalism).

From now on, $Q_{\rho\tau}$ denotes the $n\times n$ matrix with elements $q_{\alpha\beta;\rho\tau}$ (ρ, τ fixed) and $Q_{\rho\tau}^{(k)}$ denotes the $n\times n$ matrix with elements $q_{\alpha\beta;\rho\tau}^k$ (ρ, τ fixed).

Similarly, $\Lambda_{\rho\tau}$ $[\Lambda_{\rho\tau}^{(k)}]$ denotes the $n \times n$ matrix with elements $\lambda_{\alpha\beta;\rho\tau}$ $(\lambda_{\alpha\beta;\rho\tau}^{k})$. The matrices $\Lambda_{\rho\tau}$ and $Q_{\rho\tau}$ are defined by the saddle-point equations

$$\frac{\partial G}{\partial \lambda_{\alpha\beta;\rho\tau}} = 0, \qquad \frac{\partial G}{\partial q_{\alpha\beta;\rho\tau}} = 0 \qquad \text{as} \quad r \to 0 \tag{2.9}$$

Evaluating these equations yields

$$\lambda_{\alpha\beta;\rho\tau} = \frac{p(\Delta J)^2}{2} q_{\alpha\beta;\rho\tau}^{p-1} \qquad \text{as} \quad r \to 0 \tag{2.10}$$

$$q_{\alpha\beta;\rho\tau} = \frac{\prod_{\sigma=1}^{r} \int_{\|^\sigma \mathbf{S}\|=1} d(^\sigma\mathbf{S})\ {}^\rho S^\alpha\ {}^\tau S^\beta \exp[\frac{1}{2} \sum_{\rho,\tau=1}^{r} (^\rho\mathbf{S})^T \Lambda_{\rho\tau}\ {}^\tau\mathbf{S}]}{\prod_{\sigma=1}^{r} \int_{\|^\sigma \mathbf{S}\|=1} d(^\sigma\mathbf{S}) \exp[\frac{1}{2} \sum_{\rho,\tau=1}^{r} (^\rho\mathbf{S})^T \Lambda_{\rho\tau}\ {}^\tau\mathbf{S}]} \quad \text{as } r \to 0 \tag{2.11}$$

We see that when $\rho = \tau$, our vector normalization $\|\mathbf{S}\| = 1$ and Eq. (2.11) lead immediately to the trace condition

$$\operatorname{Tr} Q_{\rho\rho} = 1 \tag{2.12}$$

whereas we have no such restriction for the matrices $Q_{\rho\tau}$ if $\rho \neq \tau$. For $n = 1$, the order parameter equations (2.10)–(2.11) reduce to the Gross and Mézard result.[6–8]

3. PROBABILITY DISTRIBUTIONS $P_{\alpha\beta}$ AND $W_{\alpha\beta}$ FOR THE OVERLAP AND SELF-CORRELATION OF THE PURE STATES OF THE SYSTEM

Parisi discovered that in the replica theory for the Ising model the parameters $Q_{\rho\tau}$ with $\rho \neq \tau$ (just numbers in this case) can be related physically to order parameters which describe the overlap between two pure states of the model.[21,22] The overlap between two pure states t and u of an Ising system with a fixed configuration $\mathcal{T}$ of random bonds is defined as

$$q(t, u; \mathcal{T}) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle S_i \rangle_{t;\mathcal{T}} \langle S_i \rangle_{u;\mathcal{T}} \tag{3.1}$$

$\langle \cdot \rangle_{t;\mathcal{T}}$ represents the thermal average restricted to the pure state t, corresponding to the fixed configuration $\mathcal{T}$. On the other hand, the parameters $Q_{\rho\rho}$ for $n = 1$ are simply equal to one and do not constitute variables of the system.

For $n > 1$, the definition (3.1) for the overlap between pure states has to be generalized. Furthermore, we have to introduce a new parameter which we shall call the *self-correlation* of a pure state. We shall prove that the generalized overlap parameter can be related to the matrices $Q_{\rho\tau}$ $(\rho \neq \tau)$, while the new parameter for self-correlation of the pure states can be related to the matrices $Q_{\rho\rho}$ in our replica formalism. For $n > 1$, these matrices $Q_{\rho\rho}$ are not completely determined by the normalization condition $\mathrm{Tr}\, Q_{\rho\rho} = 1$ and therefore do constitute variables for the system.

The expression (3.1) is generalized to $n > 1$ by defining the following overlap between the vector components α in the pure state t and the vector components β in the pure state u, for an n-vector system with a fixed configuration $\mathscr{T}$ of random bonds:

$$q_{\alpha\beta}(t, u; \mathscr{T}) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle S_i^\alpha \rangle_{t;\mathscr{T}} \langle S_i^\beta \rangle_{u;\mathscr{T}} \tag{3.2}$$

The self-correlation of a pure state, on the other hand, is defined by means of the correlation function for the vector components at one site $\langle S_i^\alpha S_i^\beta \rangle_{t;\mathscr{T}}$ as

$$d_{\alpha\beta}(t; \mathscr{T}) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle S_i^\alpha S_i^\beta \rangle_{t;\mathscr{T}} \tag{3.3}$$

It is also possible to give definitions for overlap and self-correlation which, though intuitive, average out some of the information about the pure states contained in the general definitions (3.2) and (3.3):

$$q(t, u; \mathscr{T}) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \mathbf{S}_i^T \rangle_{t;\mathscr{T}} \langle \mathbf{S}_i \rangle_{u;\mathscr{T}} = \sum_{\alpha=1}^{n} q_{\alpha\alpha}(t, u; \mathscr{T}) \tag{3.4}$$

$$d(t; \mathscr{T}) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \mathbf{S}_i^T \mathbf{S}_i \rangle_{t;\mathscr{T}} = 1 \tag{3.5}$$

Both sets of definitions, (3.2)–(3.3) and (3.4)–(3.5), reduce to the definition (3.1) for $n = 1$. The second set of definitions requires only a parameter for overlap since the self-correlation parameter (3.5) is fixed by our normalization condition for n-vectors. We shall see below that $q_{\alpha\beta}$ and $d_{\alpha\beta}$ are related, respectively, to the matrices $Q_{\rho\tau}$ $(\rho \neq \tau)$ and $Q_{\rho\rho}$ of the replica formalism. On the other hand q and d, are related to $\mathrm{Tr}\, Q_{\rho\tau}$ $(\rho \neq \tau)$ and $\mathrm{Tr}\, Q_{\rho\rho}$, respectively.

Since the pure states of a spin glass are not related by any apparent symmetry, they cannot be extracted by means of an external magnetic field. A magnetic field which prepares a pure state would have to be site dependent and follow the local spontaneous magnetizations. We have to know these local spontaneous magnetizations before we can define such a field.

However, it is possible to obtain a description of the space of pure states by means of the probability distributions (3.8)–(3.11) for the overlaps $q_{\alpha\beta}$, q and self-correlations $d_{\alpha\beta}$, d. These probability distributions can be expressed in terms of the matrices $Q_{\rho\tau}$ from our replica formalism without recourse to an external field since it is possible to characterize pure states by the vanishing of the connected correlation functions (clustering),[23,24]

$$\langle S_{i_1}^{\alpha_1} \cdots S_{i_l}^{\alpha_l} \rangle_{t;\mathscr{T}} - \langle S_{i_1}^{\alpha_1} \rangle_{t;\mathscr{T}} \cdots \langle S_{i_l}^{\alpha_l} \rangle_{t;\mathscr{T}} = 0 \tag{3.6}$$

$$\langle S_{i_1}^{\alpha} S_{i_1}^{\beta} \cdots S_{i_l}^{\alpha} S_{i_l}^{\beta} \rangle_{t;\mathscr{T}} - \langle S_{i_1}^{\alpha} S_{i_1}^{\beta} \rangle_{t;\mathscr{T}} \cdots \langle S_{i_l}^{\alpha} S_{i_l}^{\beta} \rangle_{t;\mathscr{T}} = 0 \tag{3.7}$$

By following Parisi[21,22] and De Dominicis and Young,[25] we establish the relation between the distributions (3.8)–(3.11) and our replica formalism.

We now merely define the distributions and state the results. For a fixed configuration of random bonds $\mathscr{T}$, the probability distributions for the overlaps $q_{\alpha\beta}$, q and the self-correlations $d_{\alpha\beta}$, d defined in Eqs. (3.2)–(3.5) are given by

$$\begin{aligned} P_{\alpha\beta;\mathscr{T}}(q_{\alpha\beta}) &= \sum_{t,u=1}^{K} P_{t;\mathscr{T}} P_{u;\mathscr{T}}\, \delta[q_{\alpha\beta} - q_{\alpha\beta}(t, u; \mathscr{T})] \\ W_{\alpha\beta;\mathscr{T}}(d_{\alpha\beta}) &= \sum_{t=1}^{K} P_{t;\mathscr{T}}\, \delta[d_{\alpha\beta} - d_{\alpha\beta}(t; \mathscr{T})] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} P_{\mathscr{T}}(q) &= \sum_{t,u=1}^{K} P_{t;\mathscr{T}} P_{u;\mathscr{T}}\, \delta[q - q(t, u; \mathscr{T})] \\ W_{\mathscr{T}}(d) &= \sum_{t=1}^{K} P_{t;\mathscr{T}}\, \delta(d-1) = \delta(d-1) \end{aligned} \tag{3.9}$$

The averages of these distributions over the random couplings become

$$\begin{aligned} P_{\alpha\beta}(q_{\alpha\beta}) &\equiv \overline{P_{\alpha\beta;\mathscr{T}}(q_{\alpha\beta})} \\ W_{\alpha\beta}(d_{\alpha\beta}) &\equiv \overline{W_{\alpha\beta;\mathscr{T}}(d_{\alpha\beta})} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} P(q) &\equiv \overline{P_{\mathscr{T}}(q)} \\ W(d) &\equiv \overline{W_{\mathscr{T}}(d)} = \delta(d-1) \end{aligned} \tag{3.11}$$

Even in the case of the nonrandom n-vector model (where these probability distributions can be defined in an analogous fashion), $P_{\alpha\beta}$, $W_{\alpha\beta}$, and

P are nontrivial if $n > 1$. For $p = 2$, they can be expressed analytically and for general n in terms of hypergeometric functions ${}_2F_1$, or alternatively, associated Legendre functions of the second kind Q_ν.[27] We shall see in the next section that the distributions $P_{\alpha\beta}$, $W_{\alpha\beta}$, and P for the nonrandom model describe the geometrical degeneracy of the matrices $Q_{\rho\rho}$ and of the longitudinal part of the matrices $Q_{\rho\tau}$ $(\rho \neq \tau)$ for our quenched system.

As mentioned before, the probability distributions (3.10)–(3.11) can now be expressed in terms of the parameters $q_{\alpha\beta;\rho\tau}$ from the replica formalism in Section 2. In ref. 26 we show that

$$
\begin{aligned}
P_{\alpha\beta}(q_{\alpha\beta}) &= \lim_{r \to 0} \frac{1}{r(r-1)} \sum_{\rho \neq \tau} \delta(q_{\alpha\beta} - q_{\alpha\beta;\rho\tau}) \\
W_{\alpha\beta}(d_{\alpha\beta}) &= \lim_{r \to 0} \frac{1}{r} \sum_{\rho=1}^{r} \delta(d_{\alpha\beta} - q_{\alpha\beta;\rho\rho})
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
P(q) &= \lim_{r \to 0} \frac{1}{r(r-1)} \sum_{\rho \neq \tau} \delta(q - \operatorname{Tr} Q_{\rho\tau}) \\
W(d) &= \lim_{r \to 0} \frac{1}{r} \sum_{\rho=1}^{r} \delta(d - \operatorname{Tr} Q_{\rho\rho}) = \delta(d-1)
\end{aligned}
\tag{3.13}
$$

The replica formalism in the previous section indicates that a complete description of our model is only possible if we take the full set of parameters $q_{\alpha\beta;\rho\tau}$ into consideration. This eliminates the description (3.13), and the probability distributions $P_{\alpha\beta}(q_{\alpha\beta})$ and $W_{\alpha\beta}(d_{\alpha\beta})$ for the overlaps and self-correlations of the pure states of the system become the physical order parameters. However, it is still possible to give a "mean-field description" of our mean-field model by using the functions (3.13).

4. GENERALIZING THE PARISI THEORY TO $n > 1$

For $n = 1$, the replica formalism generates an order parameter matrix $\mathcal{Q} = (Q_{\rho\tau})$, where each $Q_{\rho\tau}$ is a number. For $n > 1$, the replica formalism generates an order parameter matrix $\mathcal{Q} = (q_{\alpha\beta;\rho\tau})$, i.e., an order parameter matrix $\mathcal{Q} = (Q_{\rho\tau})$, where each element $Q_{\rho\tau}$ is now an $n \times n$ matrix. This introduces two new features to the problem.

First, the diagonal elements $Q_{\rho\rho}$, which did not constitute variables of the problem for $n = 1$, become variables when $n > 1$, since the trace condition $\operatorname{Tr} Q_{\rho\rho} = 1$ no longer fixes $Q_{\rho\rho}$. While the procedure for replica symmetry breaking generalizes in a straightforward manner to $n > 1$ as far as the off-diagonal matrices $Q_{\rho\tau}$ $(\rho \neq \tau)$ are concerned, it is not *a priori* clear if and how replica symmetry should be broken with respect to the on-diagonal matrices $Q_{\rho\rho}$. In particular, the concept of ultrametricity loses its meaning. In fact, we shall prove rigorously that if replica symmetry for the off-diagonal matrices $Q_{\rho\tau}$ $(\rho \neq \tau)$ is broken according to a generalized Parisi scheme, then the matrices $Q_{\rho\rho}$ along the diagonal must be replica-symmetric.

The second problem is what form the matrices $Q_{\rho\tau}$ should have and how their form should change as replica symmetry is broken. It is useful at this point to introduce the notion of *component symmetry*. We shall say that an off-diagonal matrix $Q_{\rho\tau}$ $(\rho \neq \tau)$ is component-symmetric if all its elements are identical, and we shall say that an on-diagonal matrix $Q_{\rho\rho}$ is component-symmetric if, respectively, all its diagonal elements are identical and all its off-diagonal elements are identical. Formally,

$$\begin{aligned} q_{\alpha\beta;\rho\tau} &= q_{\rho\tau} \qquad \forall \alpha, \beta \quad \text{if} \quad \rho \neq \tau \\ q_{\alpha\beta;\rho\rho} &= q_{\rho\rho} \qquad \forall \alpha \neq \beta \\ q_{\alpha\alpha;\rho\rho} &= \frac{1}{n} \qquad \forall \alpha \end{aligned} \tag{4.1}$$

In the case of only one pure state for the system, we shall see below that we must have both component symmetry and replica symmetry. Both replica and component symmetry are thus a requirement for any high-temperature solution. At low temperatures, both component and replica symmetry must be broken. The way in which component symmetry is broken is uniquely determined apart from a geometrical degeneracy which is independent of the bonds and which corresponds to the time-reversal symmetry of the $n = 1$ model.

4.1. Form of the Matrices $Q_{\rho\tau}$

From Eq. (2.6) we see that for $p = 2$ the expression G remains unchanged if we multiply each matrix $Q_{\rho\tau}$ from the left with an orthonormal matrix O_ρ and from the right with an orthonormal matrix O_τ. This is because the Euclidean matrix norm is invariant if a matrix is multiplied by an orthonormal matrix and because of the symmetry of the integral. Thus, the solutions $Q_{\rho\tau}$ will have a degeneracy

$$Q_{\rho\tau} = O_\rho^T \tilde{Q}_{\rho\tau} O_\tau \qquad (p = 2) \tag{4.2}$$

where $\tilde{Q}_{\rho\tau}$ denotes some standard form of the matrix $Q_{\rho\tau}$. For $p > 2$, we see from Eq. (2.6) that the solutions $Q_{\rho\tau}$ can only have a restricted degeneracy compared to $p=2$, with the orthonormal matrices O_ρ being replaced by permutation matrices P_ρ:

$$Q_{\rho\tau}=P_\rho^T\tilde{Q}_{\rho\tau}P_\tau \qquad (p>2) \tag{4.3}$$

The $n!$ matrices P_ρ are defined to permute the components of a vector upon multiplication by them.

The degeneracies (4.2) and (4.3) of our solutions $Q_{\rho\tau}$ arise from the symmetry of the Hamiltonian (1.1) and are the analogue of the time-reversal symmetry we find for the $n=1$ model. The explicit form of the $Q_{\rho\tau}$ is then derived as follows.

4.1.1. $Q_{\rho\rho}$. If we consider the spectral decomposition of a 2×2 matrix, we see that the most general 2×2 matrix $Q_{\rho\rho}$ which is symmetric and has a fixed trace $\mathrm{Tr}\, Q_{\rho\rho}=1$ can be written in the following form:

$$Q_{\rho\rho}=n^{-1}(1-d_\rho)I+d_\rho\cdot\hat{\mathbf{v}}(\hat{\mathbf{v}})^T \tag{4.4}$$

where $n=2$, I is the unit matrix, $\hat{\mathbf{v}}$ is some unit vector, and d_ρ is an order parameter. This equation will only produce the required degeneracy (4.2) for $p=2$ when $\hat{\mathbf{v}}$ is arbitrary. The arbitrary unit vectors $\hat{\mathbf{v}}$ correspond physically to the arbitrary orientations of the magnetic field and are replaced by the Cartesian unit basis vectors $\hat{\mathbf{e}}_\alpha$ when $p>2$. Since the magnetic field and the unit basis vectors play the same physical role for $n>2$ as they do for $n=2$, we find that the most general form of the matrices $Q_{\rho\rho}$ must be given by

$$\begin{aligned} Q_{\rho\rho}&=n^{-1}(1-d_\rho)I+d_\rho\cdot{}^\rho\hat{\mathbf{s}}({}^\rho\hat{\mathbf{s}})^T \qquad &(p=2)\\ Q_{\rho\rho}&=n^{-1}(1-d_\rho)I+d_\rho\cdot\hat{\mathbf{e}}_{f(\rho)}(\hat{\mathbf{e}}_{f(\rho)})^T \qquad &(p>2)\end{aligned} \tag{4.5}$$

where the ${}^\rho\hat{\mathbf{s}}$ represent arbitrary unit vectors and where the function $f(p)$ maps replica indices onto coordinate numbers α. If we extract the degeneracies (4.2) and (4.3) from the expressions (4.5), we see that the matrix $\tilde{Q}_{\rho\rho}$ is of the form

$$\tilde{Q}_{\rho\rho}=n^{-1}\,\mathrm{diag}[1+(n-1)\,d_\rho,\,1-d_\rho,\ldots,\,1-d_\rho] \tag{4.6}$$

This matrix could also have been obtained by making an ansatz for the maximum anisotropy we expect for the eigenvalue spectrum in an n-vector system, i.e., a nondegenerate (longitudinal) eigenvalue and a (transverse) eigenvalue with $(n-1)$-fold degeneracy. We shall call d_ρ the self-correlation parameter, since it determines the probability distribution (4.12) for the self-correlation of the pure states of the system. It is also called the quadrupolar deformation parameter in the literature.[11-17]

4.1.2. $Q_{\rho\tau}$ ($\rho \neq \tau$). We expect the same maximum anisotropy for the eigenvalue spectrum of $\tilde{Q}_{\rho\tau}$ when $\rho \neq \tau$. Thus,

$$\tilde{Q}_{\rho\tau} = n^{-1} \operatorname{diag}[q_{\rho\tau}^{(L)}, q_{\rho\tau}^{(T)}, ..., q_{\rho\tau}^{(T)}] \qquad (\rho \neq \tau) \tag{4.7}$$

where $q_{\rho\tau}^{(L)}$ denotes the nondegenerate (longitudinal) eigenvalue and $q_{\rho\tau}^{(T)}$ the (transverse) eigenvalue with $(n-1)$-fold degeneracy.

It is interesting to note that for $q_{\rho\tau}^{(L)} \equiv m^2$, $q_{\rho\tau}^{(T)} \equiv 0$, and $n^{-1}(1-d_\rho) \equiv 1/(2J)$ when $p=2$, and $q_{\rho\tau}^{(L)} \equiv q^2$, $q_{\rho\tau}^{(T)} \equiv 0$, $n^{-1}(1-d_\rho) \equiv 1/(pJq^{p-2})$ when $p>2$, Eqs. (4.6) and (4.7) combined with the degeneracy (4.2)–(4.3) represent the matrices describing the overlaps and self-correlations for the pure states of the nonrandom n-vector model.[27] This confirms that the degeneracy in the matrices $Q_{\rho\tau}$ is just a geometrical effect of our spin model which is independent of the bonds and any randomness. This "geometrical degeneracy" corresponds exactly to the time-reversal symmetry of the $n=1$ model, which is also independent of bonds and randomness.

4.2. The Geometrical Degeneracy of Broken Component Symmetry

Because of the degeneracy (4.2) and (4.3), it suffices to consider the solution matrix $\tilde{\mathscr{Q}} \equiv (\tilde{Q}_{\rho\tau}) \equiv (\tilde{q}_{\alpha\beta;\rho\tau})$ with $\tilde{Q}_{\rho\tau}$ given in Eqs. (4.6) and (4.7). The most general matrix $\mathscr{Q} = (Q_{\rho\tau})$ is then obtained by means of a similarity transformation

$$\mathscr{Q} = T^T \tilde{\mathscr{Q}} T \tag{4.8}$$

where T is of block diagonal form,

$$T \equiv \operatorname{blockdiag}(T_1, ..., T_k)$$
$$T_\rho = \begin{cases} O_\rho & (p=2) \\ P_\rho & (p>2) \end{cases} \tag{4.9}$$

and the matrices T_ρ are arbitrary orthonormal $n \times n$ matrices for $p=2$ and arbitrary $n \times n$ permutation matrices for $p>2$.

As we mentioned above, the geometrical degeneracy of the solutions expressed in Eq. (4.8) corresponds to the time-reversal symmetry for the $n=1$ model. The latter is incorporated into the formalism for $n=1$ if we include a field $h = \pm 1$ into the probability distribution P for the overlap of pure states

$$P(q) = \lim_{r \to 0} \frac{1}{r(r-1)} \sum_{h=\pm 1} \sum_{\rho \neq \tau} \delta(q - hQ_{\rho\tau}) \tag{4.10}$$

The field h simply makes $P(q)$ symmetrical about the ordinate, $P(q) = P(-q)$.

In the same fashion, the geometrical degeneracy (4.8) is incorporated into the probability distributions $P_{\alpha\beta}$ and $W_{\alpha\beta}$ for the overlap of pure states when $n > 1$:

$$
\begin{aligned}
P_{\alpha\beta}(q_{\alpha\beta}) &= \lim_{r \to 0} \frac{1}{r(r-1)} \sum_{\{T\}} \sum_{\rho \neq \tau} \delta(q_{\alpha\beta} - \langle\alpha|\ T_\rho^T \tilde{Q}_{\rho\tau} T_r\ |\beta\rangle) \\
W_{\alpha\beta}(d_{\alpha\beta}) &= \lim_{r \to 0} \frac{1}{r} \sum_{\{T\}} \sum_{\rho=1}^{r} \delta(d_{\alpha\beta} - \langle\alpha|\ T_\rho^T \tilde{Q}_{\rho\rho} T_\rho\ |\beta\rangle)
\end{aligned}
\tag{4.11}
$$

where for notational reasons we have temporarily written the element $M_{\alpha\beta}$ of a matrix M as $\langle\alpha|\ M\ |\beta\rangle$ and where $\sum_{\{T\}}$ denotes the sum over all possible configurations of the matrix T.

This geometrical degeneracy of $P_{\alpha\beta}$ and $W_{\alpha\beta}$ is nontrivial when $p = 2$. As we mentioned before, for $q_{\rho\tau}^{(L)} \equiv m^2$, $q_{\rho\tau}^{(T)} \equiv 0$, and $n^{-1}(1 - d_\rho) \equiv 1/(2J)$ the matrices $Q_{\rho\tau}$ describe the overlaps and self-correlations for the pure states of the nonrandom n-vector model.[27] That is, $W_{\alpha\beta}$ and $P_{\alpha\beta}$ for the nonrandom model describe precisely the geometrical degeneracy for the quenched matrices $Q_{\rho\rho}$ and the longitudinal part of $Q_{\rho\tau}$, respectively. In ref. 27 we have shown that the nonrandom $W_{\alpha\beta}$ and $P_{\alpha\beta}$ can be expressed in terms of hypergeometric functions ${}_2F_1$, or, alternatively, associated Legendre functions of the second kind Q_ν.

Since the geometrical degeneracy is simply superimposed on our standard solutions $\tilde{\mathcal{Q}} \equiv (\tilde{Q}_{\rho\tau}) \equiv (\tilde{q}_{\alpha\beta;\rho\tau})$ from Eqs. (4.6)–(4.7), we shall only be concerned with solutions of this form and the corresponding (unsymmetrized) probability distributions

$$
P_{\alpha\beta}(q_{\alpha\beta}) = \begin{cases}
\lim_{r \to 0} \frac{1}{r(r-1)} \sum_{\rho \neq \tau} \delta(q_{\alpha\beta} - q_{\rho\tau}^{(L)}) & \alpha = \beta = 1 \\
\lim_{r \to 0} \frac{1}{r(r-1)} \sum_{\rho \neq \tau} \delta(q_{\alpha\beta} - q_{\rho\tau}^{(T)}) & \alpha = \beta > 1 \\
\delta(q_{\alpha\beta}) & \text{else}
\end{cases}
$$

$$
W_{\alpha\beta}(d_{\alpha\beta}) = \begin{cases}
\lim_{r \to 0} \frac{1}{r} \sum_{\rho=1}^{r} \delta\left(d_{\alpha\beta} - \frac{1 + (n-1)\, d_\rho}{n}\right) & \alpha = \beta = 1 \\
\lim_{r \to 0} \frac{1}{r} \sum_{\rho=1}^{r} \delta\left(d_{\alpha\beta} - \frac{1 - d_\rho}{n}\right) & \alpha = \beta > 1 \\
\delta(d_{\alpha\beta}) & \alpha \neq \beta
\end{cases}
\tag{4.12}
$$

4.3. High-Temperature Solution: Replica and Component Symmetry

At high temperatures, we expect only one pure state. Because of the degeneracy (4.8) and Eq. (4.11), this is only possible if we have both replica and component symmetry. With *0* denoting the zero matrix, the only possible high-temperature solution is therefore

$$Q_{\rho\tau} = 0 \qquad \forall \rho \neq \tau$$
$$Q_{\rho\rho} = \frac{1}{n} I \tag{4.13}$$

4.4. Low-Temperature Solutions: Breaking the Replica and Component Symmetry. Longitudinal, Transverse, and Mixed Solutions

From Eqs. (4.6)–(4.8) we see that component symmetry is broken whenever we have one $q_{\rho\tau}^{(L,T)} \neq 0$ or one $d_\rho \neq 0$. The way in which it is broken is completely determined by Eqs. (4.6)–(4.7) as soon as we choose the three order parameters $q_{\rho\tau}^{(L)}$, $q_{\rho\tau}^{(T)}$, and d_ρ (modulo the geometrical degeneracy described above). This allows for longitudinal ($q_{\rho\tau}^T = 0$), transverse ($q_{\rho\tau}^{(L)} = 0$), or mixed ($q_{\rho\tau}^{(L)}, q_{\rho\tau}^{(T)} \neq 0$) low-temperature solutions. On the other hand, replica symmetry is broken if any of the parameters $q_{\rho\tau}^{(L,T)}$ and d_ρ varies between replicas.

When replica symmetry is maintained, Eqs. (4.6)–(4.7) allow for low-temperature solutions of the form

$$\tilde{Q}_{\rho\tau} = \operatorname{diag}[q^{(L)}, q^{(T)}, \ldots, q^{(T)}] \qquad (\rho \neq \tau)$$
$$\tilde{Q}_{\rho\rho} = n^{-1} \operatorname{diag}[1 + (n-1)d, 1 - d, \ldots, 1 - d] \tag{4.14}$$

However, we expect this solution to be unstable for $n > 1$, just as it is for $n = 1$. The fact that for $n > 1$ we have the far more extensive geometrical degeneracy of pure states described before replacing the time-reversal degeneracy of the $n = 1$ model is not sufficient. We still require the additional degeneracy of the pure states brought about by the random configurations of bonds. This degeneracy can only be generated by breaking the replica symmetry.

But how should the replica symmetry be broken? The Parisi recipe for $n = 1$[28,22,24] generalizes immediately to the order parameters $q_{\rho\tau}^{(L)}$, $q_{\rho\tau}^{(T)}$, which can simply be treated (as pairs) like the corresponding single parameters $q_{\rho\tau}$ for the $n = 1$ model. However, as a scheme for breaking the symmetry of a two-dimensional array of parameters $q_{\rho\tau}$, Parisi's scheme does not tell us anything about how the symmetry in the one-dimensional array of parameters d_ρ should be broken. In particular, the concept of ultrametricity loses its meaning.

One of the fundamental features of the Parisi scheme is that the maximum overlap of a replica ρ with another replica is the same for all replicas. Therefore, it stands to reason that the self-correlation should be the same for all replicas. In other words, replica symmetry for the diagonal parameters d_ρ should not be broken, even at low temperatures.

This rather intuitive argument can also be made rigorous. In ref. 26 we show by means of the Hölder inequality that if symmetry for the off-diagonal parameters $q_{\rho\tau}^{(L,T)}$ is broken by following the Parisi procedure, then the self-correlation parameters d_ρ must be replica-symmetric in order to find the dominant saddle point for G in Eq. (2.6).

Consequently, we can adopt the following recipe for finding the low-temperature solution of the matrix $\tilde{\mathcal{Q}} = (\tilde{Q}_{\rho\tau})$:

1. Replica symmetry for the off-diagonal (overlap) parameters $q_{\rho\tau}^{(L,T)}$ is broken according to the Parisi scheme for $n = 1$ by simply performing all symmetry-breaking steps on pairs of parameters $q_{\rho\tau}^{(L)}, q_{\rho\tau}^{(T)}$ rather than on a single parameter $q_{\rho\tau}$. In a more formal fashion, after k steps of symmetry breaking, we obtain a sequence of subdivisions of the set of replicas into cluster sizes

$$r \equiv m_0 \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k \geqslant m_{k+1} \equiv 1 \tag{4.15}$$

and associated with it a hierarchy of overlap parameters given by

$$q_{\rho\tau}^{(L,T)} = q_i^{(L,T)} \quad \text{if} \quad I\left(\frac{\rho}{m_i}\right) = I\left(\frac{\tau}{m_i}\right) \quad \text{and} \quad I\left(\frac{\rho}{m_{i+1}}\right) \neq I\left(\frac{\tau}{m_{i+1}}\right)$$

$$(\rho \neq \tau;\ i = 0, 1, 2, \dots, r) \tag{4.16}$$

$$0 \leqslant q_0^{(L)} \leqslant q_1^{(L)} \leqslant \cdots q_r^{(L)} \leqslant 1$$

$$0 \leqslant q_0^{(T)} \leqslant q_1^{(T)} \leqslant \cdots q_r^{(T)} \leqslant 1$$

where $I(x)$ is the smallest integer greater than or equal to x.

2. Replica symmetry for the diagonal (self-correlation) parameters d_ρ is conserved, i.e., $d_\rho = d$ for all ρ.

The actual matrices $\tilde{Q}_{\rho\tau}$ are then determined from the order parameters $q_{\rho\tau}^{(L,T)}$ and d according to Eqs. (4.6)–(4.7).

Eventually, just as for the $n = 1$ model, the procedure of breaking the replica symmetry is carried out an infinite number of times and the free energy a/kT, which was originally a discrete function of the parameters $q_i^{(L,T)}$, m_i, and d, becomes a functional of Parisi-type functions $q^{(L,T)}(x)$ and a parameter d. And the problem of maximizing a/kT with respect to the $q_i^{(L,T)}$, m_i, and d has been reduced to the variational problem of maximizing a/kT with respect to $q^{(L,T)}(x)$ and d.

In this paper, however, we shall restrict ourselves to breaking the replica symmetry in the limit $p \to \infty$, which allows the symmetry-breaking procedure to terminate after a finite number of steps.

The derivation of the free energy (2.8) after k steps of symmetry breaking is given in ref. 26. Since it involves a considerable amount of algebra, we merely state the result here, which in the limit $r \to 0$ and by setting $d_\rho = d$ becomes

$$\begin{aligned}\overline{\left(\frac{a}{kT}\right)} = & -\ln 2\pi^{(n-1)/2} + \frac{\lambda_k^{(T)}}{2} \\ & + \sum_{j=0}^{k} (m_j - m_{j+1}) \sum_{\alpha=1}^{n} \frac{q_j^{(\alpha)}}{2} \left\{ \lambda_j^{(\alpha)} - \frac{(\Delta J)^2}{2} [q_j^{(\alpha)}]^{p-1} \right\} \\ & - \frac{\mu^{(T)}(d)}{2} + \sum_{\alpha=1}^{n} \frac{w^{(\alpha)}(d)}{2} \left\{ \mu^{(\alpha)}(d) - \frac{(\Delta J)^2}{2} [w^{(\alpha)}(d)]^{p-1} \right\} \\ & - \frac{1}{m_1} \int_{-\infty}^{\infty} D\mathbf{x}_0 \ln \int_{-\infty}^{\infty} D\mathbf{x}_1 \\ & \times \left[\cdots \int_{-\infty}^{\infty} D\mathbf{x}_k \, [F(n, k, \lambda_j^{(\alpha)}, d, \mathbf{x}_j)]^{m_k/m_{k+1}} \cdots \right]^{m_1/m_2} \end{aligned} \tag{4.17}$$

Here, we have used the conventions

$$m_0 \equiv 0, \qquad m_{k+1} \equiv 1, \qquad \lambda_{-1}^{(L,T)} \equiv 0, \qquad q_j^{(\alpha)}, \lambda_j^{(\alpha)} \equiv \begin{cases} q_j^{(L)}, \lambda_j^{(L)} & \alpha = 1 \\ q_j^{(T)}, \lambda_j^{(T)} & \alpha \neq 1 \end{cases}$$

$$D\mathbf{x}_j \equiv \frac{d\mathbf{x}_j}{(2\pi)^{n/2}} e^{-\mathbf{x}_j^T \mathbf{x}_j/2} \tag{4.18}$$

and we have defined the functions

$$w^{(L)}(d) \equiv \left[\frac{1 + (n-1)d}{n}\right], \qquad w^{(T)}(d) \equiv \left(\frac{1-d}{n}\right) \tag{4.19}$$

and

$$\begin{aligned} & F[n, k, \lambda_j^{(\alpha)}, d, \mathbf{x}_j] \\ & \quad \equiv \int_{-1}^{1} dy \, (1 - y^2)^{(n-3)/2} \exp\left\{ \frac{y^2}{2} [\mu^{(L)}(d) - \mu^{(T)}(d) - \lambda_k^{(L)} + \lambda_k^{(T)}] \right. \\ & \quad \left. + y \sum_{j=0}^{k} x_j^1 (\lambda_j^{(L)} - \lambda_{j-1}^{(L)})^{1/2} \right\} \\ & \quad \times \frac{I_{(n-3)/2}[\|\sum_{j=0}^{k} (\lambda_j^{(T)} - \lambda_{j-1}^{(T)})^{1/2} \mathbf{x}_j\|_{n-1} (1 - y^2)^{1/2}]}{[\frac{1}{2} \|\sum_{j=0}^{k} (\lambda_j^{(T)} - \lambda_{j-1}^{(T)})^{1/2} \mathbf{x}_j\|_{n-1} (1 - y^2)^{1/2}]^{(n-3)/2}} \end{aligned} \tag{4.20}$$

with $\|\mathbf{a}\|_{n-1} \equiv [\sum_{\alpha=2}^{n} (a^{\alpha})^2]^{1/2}$ and I_ν denoting modified Bessel functions of order ν.

Finally, the order parameter equations (2.10) become

$$\lambda_j^{(L,T)} = \frac{p(\Delta J)^2}{2} [q_j^{(L,T)}]^{p-1}$$
$$\mu^{(L,T)}(d) = \frac{p(\Delta J)^2}{2} [w^{(L,T)}(d)]^{p-1} \tag{4.21}$$

5. REPLICA-SYMMETRIC SOLUTIONS AS $p \to \infty$

5.1. Replica-Symmetric Free Energy and Order Parameter Equations for General p

By setting $k=0$, $m_1=1$, $q_0^{(L,T)} \equiv q^{(L,T)}$, and $\lambda_0^{(L,T)} \equiv \lambda^{(L,T)}$ in Eq. (4.17), we obtain the replica-symmetric free energy per spin

$$\overline{\left(\frac{a}{kT}\right)} = -\ln 2\pi^{(n-1)/2} + \frac{\lambda^{(T)}}{2} - \sum_{\alpha=1}^{n} \frac{q^{(\alpha)}}{2} \left\{ \lambda^{(\alpha)} - \frac{(\Delta J)^2}{2} [q^{(\alpha)}]^{p-1} \right\} - \frac{\mu^{(T)}}{2}$$
$$+ \sum_{\alpha=1}^{n} \frac{w^{(\alpha)}}{2} \left\{ \mu^{(\alpha)} - \frac{(\Delta J)^2}{2} [w^{(\alpha)}]^{p-1} \right\}$$
$$- \int_{-\infty}^{\infty} Dx \int_0^{\infty} \hat{D}R \ln \int_{-1}^{1} dy\, \varrho_0(x, y, R) \tag{5.1}$$

where

$$\varrho_0(x, y, R) \equiv (1-y^2)^{(n-3)/2} \exp\left\{ \frac{y^2}{2} [\mu^{(L)} - \mu^{(T)} - \lambda^{(L)} + \lambda^{(T)}] + yx(\lambda^{(L)})^{1/2} \right\} \frac{I_{(n-3)/2}[R(\lambda^{(T)})^{1/2}(1-y^2)^{1/2}]}{[\frac{1}{2}R(\lambda^{(T)})^{1/2}(1-y^2)^{1/2}]^{(n-3)/2}} \tag{5.2}$$

and

$$\hat{D}R \equiv \frac{e^{-R^2/2} R^{n-2}}{\Gamma((n-1)/2)\, 2^{(n-3)/2}}\, dR \tag{5.3}$$

Variation of the free energy (5.1) with respect to $q^{(L,T)}$ and $\mu^{(L,T)}$ simply recovers the order parameter equations (4.21) for the replica-symmetric case

$$\lambda^{(L,T)} = \frac{p(\Delta J)^2}{2} [q^{(L,T)}]^{p-1}$$
$$\mu^{(L,T)}(d) = \frac{p(\Delta J)^2}{2} [w^{(L,T)}(d)]^{p-1} \tag{5.4}$$

Differentiating the free energy (5.1) with respect to $\mu^{(L)}$ gives the order parameter equation for the diagonal (quadrupolar) parameter d,

$$\frac{1+(n-1)d}{n}=\int_{-\infty}^{\infty} Dx \int_0^{\infty} \hat{D}R \frac{\int_{-1}^{1} dy\, y^2 \varrho_0(x, y, R)}{\int_{-1}^{1} dy\, \varrho_0(x, y, R)} \tag{5.5}$$

The variation with respect to $\mu^{(T)}$ yields nothing new. This is a manifestation of the trace condition $\operatorname{Tr} \underline{Q}_{\rho\rho}=1$, which translates into $\mu^{(L)}+(n-1)\mu^{(T)}=1$. Differentiating $\overline{a/kT}$ with respect to $\lambda^{(L)}$ gives the order parameter equation for the longitudinal parameter $q^{(L)}$,

$$q^{(L)}=\int_{-\infty}^{\infty} Dx \int_0^{\infty} \hat{D}R \left[\frac{\int_{-1}^{1} dy\, y \varrho_0(x, y, R)}{\int_{-1}^{1} dy\, \varrho_0(x, y, R)}\right]^2 \tag{5.6}$$

Finally, the variation of $\overline{a/kT}$ with regard to $\lambda^{(T)}$ produces the order parameter equation for the transverse parameter $q^{(T)}$,

$$\begin{aligned} 1-q^{(T)}&=\int_{-\infty}^{\infty} Dx \int_0^{\infty} \hat{D}R \frac{\int_{-1}^{1} dy\, g(y, R)\, \varrho_0(x, y, R)}{\int_{-1}^{1} dy\, \varrho_0(x, y, R)} \\ g(y, R)&\equiv\frac{y^2}{2}+\frac{I_{(n-1)/2}[R(\lambda^{(T)})^{1/2}(1-y^2)^{1/2}]}{I_{(n-3)/2}[R(\lambda^{(T)})^{1/2}(1-y^2)^{1/2}]} \frac{R(1-y^2)^{1/2}}{2(\lambda^{(T)})^{1/2}} \end{aligned} \tag{5.7}$$

where we have used the relation

$$\frac{\partial}{\partial z} \frac{I_\nu(z)}{z^\nu}=\frac{I_{\nu+1}}{z^\nu} \tag{5.8}$$

for modified Bessel functions.[29] Equations (5.5)–(5.7) represent the general order parameter equations (2.11) for the replica-symmetric case.

5.2. Replica-Symmetric Free Energy and Order Parameter Equations as $p \to \infty$

In the limit $p \to \infty$, the order parameter equations (5.4) can only be satisfied if $\lambda^{(\alpha)} \sim 0$ (for $q^{(\alpha)}<1$) or if $\lambda^{(\alpha)} \sim \infty$ (for $q^{(\alpha)}=1$). With the relation[29]

$$I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)} \qquad \text{for small } z \tag{5.9}$$

the order parameter equation (5.7) for $q^{(T)}$ yields

$$q^{(T)}=0 \tag{5.10}$$

when $\lambda^{(T)} \sim 0$. On the other hand, with the relation

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} \sim 1 \qquad \text{for large } z \tag{5.11}$$

Eq. (5.7) cannot be satisfied when $q^{(T)} = 1$, $\lambda^{(T)} \sim \infty$. This means that the system can only occupy a longitudinal state $q^{(T)} = 0$ for large p.

In case of $q^{(T)} = 0$, Eqs. (5.5)–(5.7) reduce to

$$q^{(L)} = \int_{-\infty}^{\infty} Dx \left[\frac{\int_{-1}^{1} dy\, y\varrho_0^{(L)}(x, y)}{\int_{-1}^{1} dy\, \varrho_0^{(L)}(x, y)} \right]^2 \tag{5.12}$$

and

$$\frac{1 + (n-1)d}{n} = \int_{-\infty}^{\infty} Dx \frac{\int_{-1}^{1} dy\, y^2\varrho_0^{(L)}(x, y)}{\int_{-1}^{1} dy\, \varrho_0^{(L)}(x, y)} \tag{5.13}$$

where

$$\varrho_0^{(L)}(x, y) \equiv (1 - y^2)^{(n-3)/2} \exp\left\{ \frac{y^2}{2} [\mu^{(L)} - \mu^{(T)} - \lambda^{(L)}] + yx(\lambda^{(L)})^{1/2} \right\} \tag{5.14}$$

Combined with the result that we can only have a longitudinal solution $q^{(T)} = 0$, Eqs. (5.4) (for $q^{(T)} = 0$) and (5.12)–(5.13) represent the order parameter equations of our model when p is large. The corresponding free energy from Eq. (5.1) becomes

$$\begin{aligned} \overline{\left(\frac{a}{kT}\right)} = & -\ln \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} - \frac{q^{(L)}}{2} \left\{ \lambda^{(L)} - \frac{(\Delta J)^2}{2} [q^{(L)}]^{p-1} \right\} - \frac{\mu^{(T)}}{2} \\ & + \sum_{\alpha=1}^{n} \frac{w^{(\alpha)}}{2} \left\{ \mu^{(\alpha)} - \frac{(\Delta J)^2}{2} [w^{(\alpha)}]^{p-1} \right\} \\ & - \int_{-\infty}^{\infty} Dx \ln \int_{-1}^{1} dy\, \varrho_0^{(L)}(x, y) \end{aligned} \tag{5.15}$$

5.3. High-Temperature Solution and Transitions to Low-Temperature Phases as $p \to \infty$

The simplest possible solution of Eqs. (5.4) and (5.12)–(5.13) is

$$q^{(L)} = d = 0 \tag{5.16}$$

This represents the high-temperature solution (4.13). The (unsymmetrized) probability distributions (4.12) for the overlaps and self-correlations of the pure states of the system are obtained immediately as

$$\begin{aligned} P_{\alpha\beta}(q_{\alpha\beta}) &= \delta(q_{\alpha\beta}) \\ W_{\alpha\beta}(d_{\alpha\beta}) &= \begin{cases} \delta(d_{\alpha\beta} - 1/n) & \alpha = \beta \\ \delta(d_{\alpha\beta}) & \alpha \neq \beta \end{cases} \end{aligned} \tag{5.17}$$

in agreement with the fact that we have only one pure state. The corresponding free energy from Eq. (5.15) becomes

$$\left(\frac{\overline{a}}{kT}\right) = -\frac{(\Delta J)^2}{4n^{p-1}} - \ln \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{5.18}$$

We note that this high-temperature free energy holds for arbitrary p as well. The additive constant term is a result of our spin density $d\mathbf{S}$ and could be made to vanish by simply choosing the spin density $d\mathbf{S}/[2\pi^{n/2}/\Gamma(n/2)]$ instead. It is of no physical relevance since it leaves the thermodynamics of the system unchanged. The choice of spin density expresses how many states we count on the unit sphere. In particular, this means that the temperature where the corresponding high-temperature entropy per spin

$$\bar{s} = -\frac{(\Delta\tilde{J})^2}{4n^{p-1}kT^2} + k \ln \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{5.19}$$

becomes negative is not a critical temperature for the system as it is for $n = 1$. The $\Delta\tilde{J}$ was defined in Eq. (1.2).

At low temperatures, at least one of the parameters d or $q^{(L)}$ must be greater than zero. By expanding the right-hand side of Eq. (5.12) for small values of d and $q^{(L)}$, we find

$$q^{(L)} = \frac{p(\Delta J)^2}{2n^2} [q^{(L)}]^{p-1} + O\{[q^{(L)}]^{p-1} d\} + O\{[q^{(L)}]^{2(p-1)}\} \tag{5.20}$$

For $p > 2$, this equation for small values of $q^{(L)}$ cannot be satisfied. Thus, the order parameter $q^{(L)}$ for large values of p must display a jump discontinuity at a certain transition temperature T_q from the high-temperature phase with $q^{(L)} = 0$ to a low-temperature state with $q^{(L)} \geqslant \varepsilon > 0$. The temperature T_q has to be obtained numerically from Eq. (5.12) when $p > 2$.

We note that for $p = 2$ Eq. (5.20) yields the phase transition point $\Delta J_q = n$ in agreement with refs. 9–17 (if we take the spin normalization $\|\mathbf{S}\| = \sqrt{n}$ into account). Since (5.12) corresponds to a longitudinal solution, this shows that ΔJ_q is independent of the particular low-temperature ansatz, as one might expect from the argument that ΔJ_q corresponds to the temperature where "off-diagonal" fluctuations of the most general matrix $\mathcal{Q}$ away from the high-temperature solution are allowed.

Expanding the right-hand side of Eq. (5.13) for small values of d and $q^{(L)}$, on the other hand, gives

$$d = \frac{p(p-1)(\Delta J)^2}{2n^{p-1}(n+2)} d + O(d^2) + O\{d[q^{(L)}]^{p-1}\} + O\{[q^{(L)}]^{2(p-1)}\} \tag{5.21}$$

This yields the transition point ΔJ_d from the high-temperature phase with $d=0$ to a low-temperature state with $d>0$:

$$\Delta J_d = \left[\frac{2n^{p-1}(n+2)}{p(p-1)}\right]^{1/2} \tag{5.22}$$

For large p, this means that a continuous transition to a spin-glass state with $d>0$ is only possible as $T \to 0$. For all finite spin-glass transition temperatures T_g and for large p we must therefore have a jump discontinuity in the self-correlation (quadrupolar) parameter d as well. This will be confirmed by our findings in Section 6.

Since we do not expect any replica-symmetric low-temperature solution to be physical for the reasons listed in Section 4, we shall not dwell on their investigation here, but rather proceed to breaking the replica symmetry in the next section. We conclude by proving the stability of the high-temperature solution.

5.4. Stability of the High-Temperature Solution as $p \to \infty$

The free energy $\overline{a/kT}$ given by Eqs. (2.6) and (2.8) will be stable with respect to fluctuations of the general solution matrix $\mathcal{Q} = (q_{\alpha\beta;\rho\tau})$ about its equilibrium configuration if the Hessian of $\overline{a/kT}$ with respect to $q_{\alpha\beta;\rho\tau}$ is positive-definite. The auxiliary parameters $\lambda_{\alpha\beta;\rho\tau}$ have to be expressed in this connection in terms of the physical parameters $q_{\alpha\beta;\rho\tau}$ by means of Eq. (2.10). Because of the geometrical degeneracy described in Section 4, it suffices to consider fluctuations of the standard matrix $\tilde{\mathcal{Q}} \equiv (\tilde{Q}_{\rho\tau})$ with diagonal $n \times n$ matrices $\tilde{Q}_{\rho\tau}$ given by Eqs. (4.6)–(4.7). For large p, it then suffices further to restrict fluctuations to the longitudinal ansatz

$$\begin{aligned} \tilde{Q}_{\rho\tau} &= n^{-1}\,\mathrm{diag}[q^{(L)}_{\rho\tau}, 0,..., 0] \qquad (\rho \neq \tau) \\ \tilde{Q}_{\rho\rho} &= n^{-1}\,\mathrm{diag}[1+(n-1)\,d_\rho,\, 1-d_\rho,...,\, 1-d_\rho] \end{aligned} \tag{5.23}$$

After inserting this longitudinal ansatz combined with the relation (2.10) into Eq. (2.6), the stationarity of the free energy (2.8) depends only on fluctuations $\eta_{\rho\tau}$ of the off-diagonal parameters $q^{(L)}_{\rho\tau}$ and on fluctuations ε_ρ of the diagonal parameters d_ρ. Thus, the situation is completely analogous to the stability analysis performed by de Almeida and Thouless[20] for the Sherrington–Kirkpatrick model.

In ref. 26 we have shown by means of the Hölder inequality that the free energy can only be stationary if all the diagonal parameters d_ρ are equal to some d. Since at high temperatures d is uniquely determined

($d=0$), the high-temperature solution will then be stable with respect to arbitrary fluctuations ε_ρ of d_ρ.

It remains to be shown that the solution is stable with regard to fluctuations of the off-diagonal parameters $q_{\rho\tau}^{(L)}$, i.e., with respect to fluctuations which lead to replica symmetry breaking. By following ref. 20, one shows that the eigenvalue of the Hessian of the free energy (2.8) corresponding to these fluctuations is given by

$$EV \equiv -\frac{\partial^2 G}{\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\rho\tau}^{(L)}} + 2\frac{\partial^2 G}{\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\rho\sigma}^{(L)}} - \frac{\partial^2 G}{\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\sigma\nu}^{(L)}} \tag{5.24}$$

where $\rho \neq \tau \neq \sigma \neq \nu$, G is defined in (2.6), and we have expressed all parameters $q_{\rho\tau}^{(L)}$ in terms of $\lambda_{\rho\tau}^{(L)}$ by means of Eq. (2.10). EV is independent of the particular realization of the replica indices because of the symmetry of the system. In order to evaluate EV for the replica-symmetric solution $q_{\rho\tau}^{(L)} = q^{(L)}$, we require the following expectation values for the spin components:

$$\begin{aligned}
\overline{\langle {}^\rho S^1\, {}^\tau S^1 \rangle} &= q^{(L)} = \int_{-\infty}^{\infty} Dx \left[\frac{\int_{-1}^{1} dy\, y\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]^2 \\
\overline{\langle {}^\rho S^1\, {}^\tau S^1\, {}^\rho S^1\, {}^\tau S^1 \rangle} &= \int_{-\infty}^{\infty} Dx \left[\frac{\int_{-1}^{1} dy\, y^2\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]^2 \\
\overline{\langle {}^\rho S^1\, {}^\tau S^1\, {}^\rho S^1\, {}^\sigma S^1 \rangle} &= \int_{-\infty}^{\infty} Dx \left[\frac{\int_{-1}^{1} dy\, y^2\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]\left[\frac{\int_{-1}^{1} dy\, y\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]^2 \\
\overline{\langle {}^\rho S^1\, {}^\tau S^1\, {}^\sigma S^1\, {}^\nu S^1 \rangle} &= \int_{-\infty}^{\infty} Dx \left[\frac{\int_{-1}^{1} dy\, y\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]^4
\end{aligned} \tag{5.25}$$

where $\varrho_0^{(L)}$ has been defined in (5.14). From Eqs. (2.6), (5.24), and (5.25), the stability condition $EV>0$ for our replica-symmetric (longitudinal) solution then becomes

$$\frac{q^{(L)}}{(p-1)\,\lambda^{(L)}} - \int_{-\infty}^{\infty} Dx \left\{\frac{\int_{-1}^{1} dy\, y^2\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}} - \left[\frac{\int_{-1}^{1} dy\, y\varrho_0^{(L)}}{\int_{-1}^{1} dy\, \varrho_0^{(L)}}\right]^2\right\}^2 > 0 \tag{5.26}$$

For $p>2$, the first term on the left diverges for the high-temperature solution $q^{(L)}=0$, while the integral is always finite. Thus, the high-temperature solution for large p is stable at all temperatures. This is analogous to the result for the $n=1$ model found by Gardner.[7]

Of course, for finite (not too large) $p>2$ we should have to investigate the stability of the high-temperature solution with regard to formation of transverse or mixed spin glasses as well. This is why Eq. (5.26) can only represent the stability condition for large p. However, we believe that the same divergent term will make the high-temperature solution stable for finite (not too large) $p > 2$ as well.

6. BREAKING REPLICA SYMMETRY AS $p \to \infty$

6.1. Solution

Let us consider the first step of symmetry breaking. By setting $k = 1$, $m_1 = m$, and $m_2 = 1$ in Eq. (4.17) we obtain the free energy

$$\overline{\left(\frac{a}{kT}\right)} = -\ln 2\pi^{(n-1)/2} + \frac{\lambda_1^{(T)}}{2} - m \sum_{\alpha=1}^{n} \frac{q_0^{(\alpha)}}{2} \left\{\lambda_0^{(\alpha)} - \frac{(\Delta J)^2}{2} [q_0^{(\alpha)}]^{p-1}\right\}$$
$$- (1-m) \sum_{\alpha=1}^{n} \frac{q_1^{(\alpha)}}{2} \left\{\lambda_1^{(\alpha)} - \frac{(\Delta J)^2}{2} [q_1^{(\alpha)}]^{p-1}\right\}$$
$$- \frac{\mu^{(T)}(d)}{2} + \sum_{\alpha=1}^{n} \frac{w^{(\alpha)}(d)}{2} \left\{\mu^{(\alpha)}(d) - \frac{(\Delta J)^2}{2} [w^{(\alpha)}(d)]^{p-1}\right\}$$
$$- \frac{1}{m} \int_{-\infty}^{\infty} D\mathbf{x}_0 \ln \int_{-\infty}^{\infty} D\mathbf{x}_1\, F^m \tag{6.1}$$

with

$$F \equiv \int_{-1}^{1} dy\, (1-y^2)^{(n-3)/2} \exp\left\{\frac{y^2}{2} [\mu^{(L)} - \mu^{(T)} - \lambda_1^{(L)} + \lambda_1^{(T)}]\right.$$
$$\left. + y[x_0^1(\lambda_0^{(L)})^{1/2} + x_1^1(\lambda_1^{(L)} - \lambda_0^{(L)})^{1/2}]\right\}$$
$$\times \frac{I_{(n-3)/2}[\|\mathbf{x}_0(\lambda_0^{(T)})^{1/2} + \mathbf{x}_1(\lambda_1^{(T)} - \lambda_0^{(T)})^{1/2}\|_{n-1} (1-y^2)^{1/2}]}{[\frac{1}{2} \|\mathbf{x}_0(\lambda_0^{(T)})^{1/2} + \mathbf{x}_1(\lambda_1^{(T)} - \lambda_0^{(T)})^{1/2}\|_{n-1} (1-y^2)^{1/2}]^{(n-3)/2}} \tag{6.2}$$

In the limit $p \to \infty$, the order parameter equations (4.21) can only be satisfied if $q_j^{(\alpha)} < 1$ and $\lambda_j^{(\alpha)} \sim 0$ or if $q_j^{(\alpha)} = 1$ and $\lambda_j^{(\alpha)} \sim \infty$. Let us assume that $\lambda_0^{(T)} \sim 0$ and $\lambda_1^{(T)} \sim \infty$. By using the relation[29]

$$I_\nu(z) \sim e^z/(2\pi z)^{1/2} \qquad \text{as} \quad z \to \infty \tag{6.3}$$

the variation of the free energy (6.1) with respect to $\lambda_1^{(T)}$ then yields $q_1^{(T)} = 0$, in contradiction to our initial assumption $q_1^{(T)} = 1$. The same happens if we assume $\lambda_0^{(T)}, \lambda_1^{(T)} \sim \infty$. Thus, we must have $\lambda_0^{(T)}, \lambda_1^{(T)} \sim 0$, i.e., $q_0^{(T)} \leqslant q_1^{(T)} < 1$. Variation of the free energy (6.1) with respect to $\lambda_1^{(T)}$ by using the relation (5.9) and $\lambda_0^{(T)}, \lambda_1^{(T)} \sim 0$ gives

$$q_0^{(T)} = q_1^{(T)} = 0 \tag{6.4}$$

Just as in the replica-symmetric case, we find therefore that the system for large p can only occupy longitudinal low-temperature states after the first step of symmetry breaking.

In order to have nontrivial symmetry breaking, we require $q_0^{(L)} < q_1^{(L)}$. Then $\lambda_0^{(L)}, \lambda_1^{(L)} \sim 0$ simply recovers the replica-symmetric case. Hence, we have $q_0^{(L)} < q_1^{(L)} = 1$. That is, the procedure for replica symmetry breaking from Section 4 terminates after the first step. This is analogous to the $n = 1$ model considered by Gross and Mézard.[6]

The free energy (6.1) for the longitudinal solution (6.4) becomes

$$\begin{aligned}\overline{\left(\frac{a}{kT}\right)} = &- \ln \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \\ &- m \frac{g_0^{(L)}}{2} \left\{ \lambda_0^{(L)} - \frac{(\Delta J)^2}{2} [q_0^{(L)}]^{p-1} \right\} \\ &- (1-m) \frac{q_1^{(L)}}{2} \left\{ \lambda_1^{(L)} - \frac{(\Delta J)^2}{2} [q_1^{(L)}]^{p-1} \right\} \\ &+ \frac{w^{(L)}(d)}{2} \left\{ \mu^{(L)} - \frac{(\Delta J)^2}{2} [w^{(L)}(d)]^{p-1} \right\} \\ &- \frac{\mu^{(T)}}{2} + (n-1) \frac{w(d)^{(T)}}{2} \left\{ \mu^{(T)} - \frac{(\Delta J)^2}{2} [w^{(T)}(d)]^{p-1} \right\} \\ &- \frac{1}{m} \int_{-\infty}^{\infty} Dx_0 \ln \int_{-\infty}^{\infty} Dx_1\, F[n, \mu^{(L)}, \mu^{(T)}, \lambda_0^{(L)}, \lambda_1^{(L)}, x_0, x_1]^m \end{aligned} \tag{6.5}$$

with

$$w^{(L)}(d) \equiv \left[\frac{1 + (n-1)d}{n} \right], \qquad w^{(T)}(d) \equiv \left(\frac{1-d}{n} \right) \tag{6.6}$$

and the function

$$\begin{aligned} F[n, \mu^{(L)}, \mu^{(T)}, \lambda_0^{(L)}, \lambda_1^{(L)}, x_0, x_1] &\equiv \int_{-1}^{1} dy\, (1-y^2)^{(n-3)/2} \exp \left\{ \frac{y^2}{2} [\mu^{(L)} - \mu^{(T)} - \lambda_1^{(L)}] \right. \\ &\left. + y(x_0(\lambda_0^{(L)})^{1/2} + x_1(\lambda_1^{(L)} - \lambda_0^{(L)})^{1/2}) \right\} \end{aligned} \tag{6.7}$$

$\lambda_j^{(L)}$ and $\mu^{(L,T)}$ are determined by the order parameter equations (4.21),

$$\begin{aligned} \lambda_j^{(L)} &= \frac{p(\Delta J)^2}{2} [q_j^{(L)}]^{p-1} \\ \mu^{(L,T)} &= \frac{p(\Delta J)^2}{2} [w^{(L,T)}(d)]^{p-1} \end{aligned} \tag{6.8}$$

combined with our result above that $\lambda_0^{(L)} \sim 0$ and $\lambda_1^{(L)} \sim \infty$.

This allows the triple integral in Eq. (6.1) to be expanded asymptotically. First, we expand the free energy (6.1) for $\lambda_0^{(L)} \sim 0$, $\lambda_1^{(L)} \sim \infty$, and

$$\lim_{\lambda_1^{(L)} \to \infty} \frac{\mu^{(L)} - \mu^{(T)}}{\lambda_1} < 1 - m$$

Differentiating the result with respect to $\lambda_1^{(L)}$, however, yields $(1-m)\, q_1^{(L)}/2 \sim 0$, in contradiction to our initial input $q_1 = 1$. Thus, we have

$$\lim_{\lambda_1^{(L)} \to \infty} \frac{\mu^{(L)} - \mu^{(T)}}{\lambda_1^{(L)}} > (1-m) \tag{6.9}$$

This is consistent with our physical expectation that the self-correlation $w^{(L)}$ of a pure state should be at least as big as the largest overlap between two pure states given by $q_1^{(L)} = 1$, i.e., $\mu^{(L)} - \mu^{(T)} \geqslant \lambda_1^{(L)}$. Expanding the free energy (6.1) for $\lambda_0^{(L)} \sim 0$, $\lambda_1^{(L)} \sim \infty$, and the condition (6.9) gives

$$\begin{aligned}
\overline{\left(\frac{a}{kT}\right)} \sim & -\frac{n-1}{2} \ln 2\pi - \frac{1}{m} \ln 2 - m \frac{q_0^{(L)}}{2} \left\{ \lambda_0^{(L)} - \frac{(\Delta J)^2}{2} [q_0^{(L)}]^{p-1} \right\} \\
& + (1-m) \frac{\lambda_1^{(L)}}{2} - (1-m) \frac{q_1^{(L)}}{2} \left\{ \lambda_1^{(L)} - \frac{(\Delta J)^2}{2} [q_1^{(L)}]^{p-1} \right\} \\
& - \frac{\mu^{(T)}}{2} + \frac{w^{(L)}(d)}{2} \left\{ \mu^{(L)} - \frac{(\Delta J)^2}{2} [w^{(L)}(d)]^{p-1} \right\} \\
& + (n-1) \frac{w(d)^{(T)}}{2} \left\{ \mu^{(T)} - \frac{(\Delta J)^2}{2} [w^{(T)}(d)]^{p-1} \right\} \\
& + \frac{n-1}{2} \ln [\mu^{(L)} - \mu^{(T)} - (1-m)\, \lambda_1^{(L)}] \\
& + O\left\{ [\lambda_0^{(L)}]^2, \frac{1}{\mu^{(L)} - \mu^{(T)} - (1-m)\, \lambda_1^{(L)}} \right\}
\end{aligned} \tag{6.10}$$

For $n = 1$, this free energy agrees with the Gross and Mézard result.[6]

Differentiating the free energy with respect to $q_0^{(L)}$, $q_1^{(L)}$, and d simply recovers Eq. (6.8). Differentiating with respect to $\lambda_0^{(L)}$, $\lambda_1^{(L)}$, $\mu^{(L)}$, and $\mu^{(T)}$ gives

$$q_0^{(L)} = 0, \qquad q_1^{(L)} = 1, \qquad d = 1 \tag{6.11}$$

consistent with our initial assumptions.

Finally, the variation with respect to m yields

$$m^2(\Delta J)^2 = 4\ln 2 + 2(n-1)m \tag{6.12}$$

where we have used Eqs. (6.11), (6.6), and (6.8). Since $0 \leqslant m \leqslant 1$, this means that solutions exist only if

$$\Delta J \geqslant \Delta J_g \equiv [4\ln 2 + 2(n-1)]^{1/2} \tag{6.13}$$

The spin-glass transition temperature $T_g = \Delta\tilde{J}/(k\,\Delta J_g)$ agrees with the Gross and Mézard result for $n=1$ and decreases as n increases.

In contrast to the $n=1$ model, however, T_g for $n>1$ does not represent the temperature where the entropy of the high-temperature solution turns negative, as we explained in Section 5 in conjunction with Eqs. (5.18)–(5.19).

The (unsymmetrized) probability distributions (4.12) for the overlaps and self-correlations of the pure states of the system are obtained as

$$\begin{aligned} P_{\alpha\beta}(q_{\alpha\beta}) &= \begin{cases} m\delta(q_{\alpha\beta}) + (1-m)\,\delta(q_{\alpha\beta}-1) & \alpha=\beta=1 \\ \delta(q_{\alpha\beta}) & \text{else} \end{cases} \\ W_{\alpha\beta}(d_{\alpha\beta}) &= \begin{cases} \delta(d_{\alpha\beta}-1) & \alpha=\beta=1 \\ \delta(d_{\alpha\beta}) & \text{else} \end{cases} \end{aligned} \tag{6.14}$$

where m represents the solution of Eq. (6.12), noting that $0 \leqslant m \leqslant 1$:

$$m \sim \frac{n-1+[(n-1)^2+(\Delta J)^2\,4\ln 2]^{1/2}}{(\Delta J)^2} \tag{6.15}$$

The free energy is evaluated by inserting m together with Eq. (6.11) into (6.10). We find

$$\left(\frac{\overline{a}}{kT}\right) \sim -\frac{(\Delta J)^2\ln 2}{X} - \frac{X}{4} + \frac{n-1}{2}\ln X + \frac{n-1}{2}\ln\frac{p}{4\pi} + O\left(\frac{1}{p}\right) \tag{6.16}$$

with

$$X \equiv (n-1) + [(n-1)^2 + (\Delta J)^2\,4\ln 2]^{1/2} \tag{6.17}$$

For $n>1$, we see that the free energy diverges as $p \to \infty$. This is a result of the fact that for $n>1$, and due to our spin normalization $\|\mathbf{S}\| = 1$, individual spin components S_i^α are $\leqslant 1$. Each interaction term in the Hamiltonian (1.1) therefore consists of the product of p spin variables of magnitude $\leqslant 1$ and becomes itself of order e^p with $0 \leqslant \varepsilon \leqslant 1$. This results in a scaling of the free energy with p if the spin variables are on average unequal to one or zero in magnitude, i.e., if $\langle \varepsilon \rangle \neq 0, 1$. For $n=1$, this kind of scaling is not observed, since ε is always equal to one.

Unfortunately, the explicit form of scaling is difficult to establish ahead of the actual calculation. Equation (6.16) shows that for the spin-glass phase it is of order $\frac{1}{2}(n-1)\ln p$ in the free energy. This is certainly different from the scaling of the high-temperature free energy given in Eq. (5.18). However, it is precisely the scaling we find for the low-temperature solution of the annealed n-vector p-spin model. The free energy for the latter, in the limit $p \to \infty$ and by taking the normalization $\|\mathbf{S}\| = 1$ into account, is given by[1]

$$\left(\frac{a}{kT}\right)_{\text{annealed}} \begin{cases} = -\dfrac{(\Delta J)^2}{4n^{p-1}} - \ln \dfrac{2\pi^{n/2}}{\Gamma(n/2)}, & T > T_c \\ \sim -\ln 2\pi^{(n-1)/2} - \dfrac{(\Delta J)^2}{4} + \dfrac{n-1}{2}\ln p + O\left(\dfrac{1}{p}\right), & T < T_c \end{cases} \tag{6.18}$$

Because of Eqs. (2.3) and (2.8) for the quenched model and the corresponding equations for the annealed model, it is always possible to obtain a finite low-temperature free energy per spin by performing the following scaling transformation:

$$\Delta J = \frac{\Delta J_0}{p^{p(n-1)/2}}, \qquad \|\mathbf{S}\| = p^{(n-1)/2} \tag{6.19}$$

as $p \to \infty$. We assume ΔJ_0 to be an intensive quantity. This scaling transformation simply adds a constant $-\frac{1}{2}(n-1)\ln p$ to the free energy, which therefore cancels the divergent term in both the quenched and annealed low-temperature solutions. However, the same scaling transformation which makes the low-temperature free energies per spin finite makes the high-temperature free energies diverge. In other words, for $n > 1$ there is not a universal scaling which could be introduced right from the start into the formalism and which would keep both the high- and low-temperature free energies per spin simultaneously finite as $p \to \infty$. In this sense, an anomalous feature emerges from the model (1.1) with respect to scaling at $p = \infty$.

We also note that the low-temperature free energy (6.16) is larger than the high-temperature free energy (5.18) for all finite temperatures $T < T_g$. This is a consequence of the fact that in the replica formalism the free energy has to be maximized in order to find the equilibrium configuration of the system. This is in contrast to ordinary statistical mechanics (cf. the annealed case) and represents a feature of all spin-glass models, even when evaluated by some alternative method (cf. the $n=1$ model considered as the random energy model[4,5]).

Finally, we see that the free energy has a jump discontinuity at the spin-glass transition temperature T_g. This indicates a first-order transition with latent heat. It is in contrast to the spin-glass transition for the $n=1$ model as $p \to \infty$, which is only of second order with no latent heat.[6,7] Furthermore, the free energy (6.16) displays a temperature dependence for $n>1$, while the corresponding free energy for the $n=1$ model is constant at all $T<T_g$. The latter is a consequence of the $n=1$ model being equivalent to the random energy model where below T_g only one low-temperature state exists. It also makes it seem unlikely that our model for $n>1$ and $p \to \infty$ could be solved by an alternative method similar to the random energy model.

6.2. Stability of the Low-Temperature Solution as $p \to \infty$

We now investigate whether the free energy given by Eqs. (2.6) and (2.8) is stationary with respect to fluctuations of the order parameters $q_{\alpha\beta;\rho\tau}$ about the equilibrium configuration just determined by the first step of replica symmetry breaking. For large p, and by following the same arguments as in Section 5.4, it suffices to insert the longitudinal ansatz (5.23) into (2.8) and to consider fluctuations of the longitudinal parameters $q_{\rho\tau}^{(L)}$ about their equilibrium configuration $q_0^{(L)}$, $q_1^{(L)}$. Again, the auxiliary parameters $\lambda_{\alpha\beta;\rho\tau}$ are linked to the physical parameters $q_{\alpha\beta;\rho\tau}$ by Eq. (2.10).

If we express all $q_{\rho\tau}^{(L)}$ in terms of $\lambda_{\rho\tau}^{(L)}$ by means of Eq. (2.10), then the free energy (2.8) will be stable if the Hessian $-\partial^2 G/\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\sigma\nu}^{(L)}$ is positive-definite. Let us first consider a diagonal element of the Hessian with ρ, τ in different clusters:

$$-\frac{\partial^2 G}{\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\rho\tau}^{(L)}} = \frac{q_0^{(L)}}{(p-1)\,\lambda_0^{(L)}} - \text{finite expression} \tag{6.20}$$

Since we have $p>2$ and $q_0^{(L)}=0$, the first term diverges. That is, all diagonal elements of the Hessian with ρ, τ in different clusters are positive and infinite. All off-diagonal elements of the Hessian, on the other hand, are finite. This means that if we have an eigenvector of the Hessian which has a nonzero component $x_{\rho\tau}$ with ρ, τ in different clusters, then the corresponding eigenvalue must be positive and infinite.

It remains to consider eigenvectors of the Hessian which have only nonzero components $x_{\rho\tau}$ with ρ, τ in the same cluster. This reduces the eigenvalue problem for the entire Hessian to an eigenvalue problem for the Hessian submatrix of a cluster, $-\partial^2 G/\partial\lambda_{\rho\tau}^{(L)}\,\partial\lambda_{\sigma\nu}$, with ρ, τ, σ, ν in the same cluster. Because of the symmetry of the solution within a cluster, the eigenvalue problem has then become analogous to the replica-symmetric case. Thus, the condition for stability of the free energy (2.8) after the first step of symmetry breaking and for large p is given by $EV > 0$, where EV has been defined in (5.24) and where it is now understood that ρ, τ, σ, ν are all in the same cluster.

In order to evaluate EV, we require the following expectation values for spin components within the same cluster:

$$\begin{aligned}
&\overline{\langle {}^{\rho}S^1\, {}^{\tau}S^1 \rangle} = q_1^{(L)} \\
&\quad = \int_{-\infty}^{\infty} Dx_0 \frac{\int_{-\infty}^{\infty} Dx^1 \,[\int_{-1}^{1} dy\, y\varrho_1^{(L)}]^2\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^{m-2}}{\int_{-\infty}^{\infty} Dx^1\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^m} \\
&\overline{\langle {}^{\rho}S^1\, {}^{\tau}S^1\, {}^{\rho}S^1\, {}^{\tau}S^1 \rangle} \\
&\quad = \int_{-\infty}^{\infty} Dx_0 \frac{\int_{-\infty}^{\infty} Dx^1 \,[\int_{-1}^{1} dy\, y^2\varrho_1^{(L)}]^2\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^{m-2}}{\int_{-\infty}^{\infty} Dx^1\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^m} \\
&\overline{\langle {}^{\rho}S^1\, {}^{\tau}S^1\, {}^{\rho}S^1\, {}^{\sigma}S^1 \rangle} \\
&\quad = \int_{-\infty}^{\infty} Dx_0 \frac{\int_{-\infty}^{\infty} Dx^1 \,[\int_{-1}^{1} dy\, y^2\varrho_1^{(L)}][\int_{-1}^{1} dy\, y\varrho_1^{(L)}]^2\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^{m-3}}{\int_{-\infty}^{\infty} Dx^1\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^m} \\
&\overline{\langle {}^{\rho}S^1\, {}^{\tau}S^1\, {}^{\sigma}S^1\, {}^{\nu}S^1 \rangle} \\
&\quad = \int_{-\infty}^{\infty} Dx_0 \frac{\int_{-\infty}^{\infty} Dx^1 \,[\int_{-1}^{1} dy\, y\varrho_1^{(L)}]^4\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^{m-4}}{\int_{-\infty}^{\infty} Dx^1\, [\int_{-1}^{1} dy\, \varrho_1^{(L)}]^m}
\end{aligned} \tag{6.21}$$

where it is understood that $\rho \neq \tau \neq \sigma \neq \nu$ and where we have defined

$$\begin{aligned}
\varrho_1^{(L)} \equiv (1-y^2)^{(n-3)/2} \exp \Bigg\{ &\frac{y^2}{2} [\mu^{(L)} - \mu^{(T)} - \lambda_1^{(L)}] \\
&+ y(x_0(\lambda_0^{(L)})^{1/2} + x_1(\lambda_1^{(L)} - \lambda_0^{(L)})^{1/2}) \Bigg\}
\end{aligned} \tag{6.22}$$

From Eqs. (2.6), (5.24), and (6.21), EV becomes

$$\begin{aligned}
EV = &\frac{q_1^{(L)}}{(p-1)\,\lambda_1^{(L)}} - \int_{-\infty}^{\infty} Dx_0 \\
&\times \frac{\int_{-\infty}^{\infty} Dx^1 \left\{ \frac{\int_{-1}^{1} dy\, y^2 \varrho_1^{(L)}}{\int_{-1}^{1} dy\, \varrho_1^{(L)}} - \left[\frac{\int_{-1}^{1} dy\, y\varrho_1^{(L)}}{\int_{-1}^{1} dy\, \varrho_1^{(L)}} \right]^2 \right\}^2 \left[\int_{-1}^{1} dy\, \varrho_1^{(L)} \right]^m}{\int_{-\infty}^{\infty} Dx^1 \left[\int_{-1}^{1} dy\, \varrho_1^{(L)} \right]^m}
\end{aligned} \tag{6.23}$$

We showed above that for large p we have $\lambda_0^{(L)} \sim 0$ and $\lambda_1^{(L)} \sim \infty$. Equation (6.11) tells us further that for large p we have

$$z \equiv \mu^{(L)} - \mu^{(T)} - \lambda_1^{(L)} \sim 0 \tag{6.24}$$

This enables us to expand the various integrals in Eq. (6.23) asymptotically for large values of p. This has to be done up to third order to obtain the first nonvanishing contribution to EV. We find

$$\begin{aligned}
&\int_{-1}^{1} dy\, \varrho_1^{(L)} \\
&\quad \sim X\left[1 - \frac{(n-1)(n-3)}{8[z+|x_1|(\lambda_1^{(L)})^{1/2}]} + \frac{(n+1)(n-1)(n-3)(n-5)}{2^7[z+|x_1|(\lambda_1^{(L)})^{1/2}]^2} + \cdots\right] \\
&\int_{-1}^{1} dy\, y\varrho_1^{(L)} \\
&\quad \sim X\left[1 - \frac{(n-1)(n+1)}{8[z+|x_1|(\lambda_1^{(L)})^{1/2}]} + \frac{(n+1)(n-1)(n^2-9)}{2^7[z+|x_1|(\lambda_1^{(L)})^{1/2}]^2} + \cdots\right] \\
&\int_{-1}^{1} dy\, y^2\varrho_1^{(L)} \\
&\quad \sim X\left[1 - \frac{(n-1)(n+5)}{8[z+|x_1|(\lambda_1^{(L)})^{1/2}]} + \frac{(n+1)(n-1)(n^2+8n-1)}{2^7[z+|x_1|(\lambda_1^{(L)})^{1/2}]^2} + \cdots\right]
\end{aligned} \tag{6.25}$$

where

$$X \equiv \frac{2^{(n-3)/2}\Gamma((n-1)/2)\exp[z/2+|x_1|(\lambda_1^{(L)})^{1/2}]}{[z+|x_1|(\lambda_1^{(L)})^{1/2}]^{(n-1)/2}} \tag{6.26}$$

By inserting the expansions (6.25) into Eq. (6.23), and using m from Eq. (6.15) and $\lambda_1^{(L)} \sim p(\Delta J)^2/2$, we find that the stability condition $EV > 0$ for the first step of replica symmetry breaking as $p \to \infty$ becomes

$$1 > \frac{\sqrt{2}(n-1)\,\Delta J}{p\{n-1+[(n-1)^2+(\Delta J)^2\, 4\ln 2]^{1/2}\}^2} \tag{6.27}$$

This condition is satisfied for all ΔJ. That is, the longitudinal solution (6.11) with one step of replica symmetry breaking is stable at all temperatures $T < T_g$ for large p.

7. DISCUSSION

The quenched model (1.1) has been investigated for the case of general n while $p \to \infty$. For $p > 2$, the model incorporates an anisotropy of type S_n

which replaces the $O(n)$ symmetry of the model for $p=2$ in the absence of a magnetic field.

In the case of Ising spin systems, the replica formalism generates an order parameter matrix $\mathscr{Q}=(Q_{\rho\tau})$, where each $Q_{\rho\tau}$ is a number. For $n>1$, the replica formalism generates an order parameter matrix $\mathscr{Q}=(Q_{\rho\tau})=(q_{\alpha\beta;\rho\tau})$, where each element $Q_{\rho\tau}$ is an $n\times n$ matrix. Interactions with $p>2$ are incorporated into the replica formalism by means of $n\times n$ Lagrange multiplier matrices $\Lambda_{\rho\tau}=(\lambda_{\alpha\beta;\rho\tau})$.

The solutions $Q_{\rho\tau}$ have a geometrical degeneracy of type $O(n)$ for $p=2$ and of type S_n for $p>2$. The most general matrix $\mathscr{Q}$ can be expressed as $\mathscr{Q}=T^T\tilde{\mathscr{Q}}T$, where $\tilde{\mathscr{Q}}\equiv(\tilde{Q}_{\rho\tau})$ represents the standard form (4.6)–(4.7) of the matrix $\mathscr{Q}$ and where T is of the block-diagonal form (4.9). This geometrical degeneracy is independent of the bonds and randomness and corresponds to the time-reversal symmetry of the $n=1$ model. For $p=2$ and $q^{(T)}=0$, the concomitant shape of the probability distributions for the overlaps and self-correlations of the pure states is given by the corresponding distributions for the nonrandom model. It can be expressed in terms of associated Legendre functions of the second kind.

The standard matrices $\tilde{Q}_{\rho\tau}$ given by Eqs. (4.6)–(4.7) are determined by three parameters. The off-diagonal matrices $\tilde{Q}_{\rho\tau}$ $(\rho\neq\tau)$ depend on the longitudinal parameter $q^{(L)}_{\rho\tau}$ and the transverse parameter $q^{(T)}_{\rho\tau}$, while the matrices $\tilde{Q}_{\rho\rho}$ depend only on the self-correlation (quadrupolar) parameter d_ρ. For $n>1$, the model is therefore described by three order parameters, $q^{(L)}_{\rho\tau}$, $q^{(T)}_{\rho\tau}$, and d_ρ.

The physical interpretation of the replica formalism shows that the off-diagonal matrices $Q_{\rho\tau}$ $(\rho\neq\tau)$ (and hence the parameters $q^{(L,T)}_{\rho\tau}$) describe the overlap (3.2) between the pure states of the system in analogy to the $n=1$ model. The matrices $Q_{\rho\rho}$ (and hence the parameters d_ρ), on the other hand, describe the "self-correlation" (3.3) for the pure states of the system. The probability distributions $P_{\alpha\beta}$ for the overlaps between the pure states and the probability distributions $W_{\alpha\beta}$ for the self-correlations of the pure states constitute the physical order parameters for the system. They are given in terms of the parameters $q^{(L,T)}_{\rho\tau}$ and d_ρ from the replica formalism in Eq. (4.12).

It is also possible to give "averaged" definitions (3.4)–(3.5) for the overlap and self-correlation of pure states which eliminate the need for an order parameter for self-correlation. Within our replica formalism, this corresponds to describing the space of pure states by the traces of the matrices $Q_{\rho\tau}$ rather than by the matrices themselves. Since $\mathrm{Tr}\,Q_{\rho\rho}=1$, d_ρ becomes invisible. Such a description would represents a mean-field theory for the model.

In addition to the concept of replica symmetry, we have introduced the notion of "component symetry" for the matrices $Q_{\rho\tau}$. At high temperatures both replica and component symmetry must be conserved. The only possible solution is given by Eq. (4.13). At low temperatures, component symmetry must be broken. In the limit of large p, we find that only longitudinal low-temperature solutions ($q_{\rho\tau}^{(T)} = 0$) are possible, both when replica symmetry is conserved and when it is broken. This is in constrast to $p = 2$ (see Introduction), where in the presence of a magnetic field or some anisotropic interaction longitudinal, transverse ($q_{\rho\tau}^{(L)} = 0$), and mixed spin-glass states ($q_{\rho\tau}^{(L)}, q_{\rho\tau}^{(T)} \neq 0$) are possible.

Even though replica-symmetric low-temperature solutions exist, our physical interpretation of the formalism tells us that replica symmetry must be broken at low temperatures. Replica symmetry for the overlap parameters $q_{\rho\tau}^{(L,T)}$ is broken according to the Parisi scheme just as for $n = 1$, while we find that replica symmetry must be conserved for the self-correlation parameters d_ρ, i.e., $d_\rho = d$ for all ρ. This can be shown by means of the Hölder inequality and is a consequence of the way in which the Parisi scheme breaks replica symmetry for the off-diagonal parameters $q_{\rho\tau}^{(L,T)}$. It does not necessarily hold for other schemes of symmetry breaking (in particular, if the ultrametric structure is missing). As $p \to \infty$, the procedure for symmetry breaking terminates after the first step. This is analogous to the Gross and Mézard result for the Ising case.[6]

The high-temperature solution is given by Eq. (4.13), and the corresponding free energy and entropy by Eqs. (5.18)–(5.19). In contrast to the Ising case, the temperature where the entropy becomes negative has no physical meaning. It is simply a consequence of how many states we count on the unit sphere and can be shifted by adopting a different convention for the spin density $d\mathrm{S}$. While the counting procedure is uniquely determined for discrete Ising spin systems, this is not the case for continuous spin systems. For large p, the high-temperature solution is stable at all temperatures down to $T = 0$. Again, this is analogous to the $n = 1$ case,[7] but is in contrast to the $p = 2$ model, where the high-temperature solution becomes unstable at the AT ($n = 1$) or the GT ($n > 1$) line.

The transition to a low-temperature phase which we expect on physical grounds is not determined by instability or an unsatisfactory negative entropy. Only longitudinal ($q^{(T)} = 0$) low-temperature solutions are possible for large p. We find that any transition to a low-temperature phase must be accompanied by a jump in both the longitudinal parameter $q_{\rho\tau}^{(L)}$ and the self-correlation parameter d for large p.

As $p \to \infty$, the system settles into the spin-glass phase (6.11) (SG1) for all $T < T_g$. The spin-glass transition temperature T_g is given by Eq. (6.13), the corresponding free energy by Eq. (6.16). For $n > 1$, the transition is of first order with a latent heat and a jump discontinuity in the order parameters $q^{(L)}$ and d. This is in contrast to the spin-glass transition for the $n=1$ model, which is only of second order.[6,7] The spin-glass solution is stable at all temperatures $T<T_g$. Since our stability analysis holds asymptotically for large values of p, the spin-glass phase SG1 will also be stable for large but finite p. This is despite the fact that for all finite p we expect at least one additional spin-glass phase for which replica symmetry is broken an infinite number of times (SG2). Thus, we have a similar situation as for the high-temperature solution. The transition to SG2 for large but finite p is not characterized by instability, but rather by a crossover line from weak to strong replica symmetry breaking. This is analogous to the crossover line from weak to strong symmetry breaking in the $p=2$ model.[14–17] However, it is in contrast to the $n=1$ model for large but finite p, where the transition from SG1 to SG2 is determined by instability.[7]

The free energy for the spin-glass phase scales as $\frac{1}{2}(n-1)\ln p$ for large p. This results from the fact that each interaction term in the Hamiltonian (1.1) consists of the product of p spin variables of magnitude $\leqslant 1$ and becomes itself of order ε^p with $0 \leqslant \varepsilon \leqslant 1$. If the spin variables are on average unequal to one or zero in magnitude, i.e., if $\langle \varepsilon \rangle \neq 0, 1$, this leads to a scaling of the free energy with p. It is explained in detail following Eq. (6.17). By performing the scaling transformation (6.19), the p dependence can be removed from the low-temperature free energy. However, there is no universal scaling which would keep the both the high- and low-temperature free energies per spin simultaneously finite as $p \to \infty$. In this sense, the model (1.1) has an anomalous behavior with respect to scaling at $p=\infty$.

Finally, the spin-glass free energy (6.16) displays a temperature dependence for $n>1$, while the corresponding free energy for the $n=1$ model is constant at all $T<T_g$. The latter is a consequence of the $n=1$ model being equivalent to the random energy model, where below T_g only one low-temperature state exists. As commented upon in Section 6, it therefore seems that our model for $n>1$ and $p \to \infty$ could not be solved by an alternative method similar to the random energy model.

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